Quantum kinematics on q-deformed quantum spaces II

Wave funtions on position and momentum space

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Abstract

The aim of Part II of this paper is to try to describe wave functions on q-deformed versions of position and momentum space. This task is done within the framework developed in Part I of the paper. In order to make Part II self-contained the most important results of Part I are reviewed. Then it is shown that q-deformed exponentials and q-deformed delta functions play the role of momentum and position eigenfunctions, respectively. Their completeness and orthonormality relations are derived. For both bases of eigenfunctions matrix elements of position and momentum operators are calculated. A q-deformed version of the spectral decomposition of multiplication operators is discussed and q-analogs of Heaviside functions are proposed. Interpreting the results from the point of view provided by the concept of quasipoints gives the formalism a physical meaning. The definition of expectation values and the calculation of probability densities are explained in detail. Finally, it is outlined how the considerations so far carry over to antisymmetrized spaces.

Keywords: Space-Time-Symmetries, Non-Commutative Geometry, Quantum Groups

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Contents

1 Introduction

As was already mentioned in Part I, deforming spacetime symmetries could yield a method to overcome the difficulties with infinities in quantum field theories [1–8]. Quantum groups and quantum spaces [9–15] seem to provide a mathematical framework for very realistic deformations. Their existence is related to of the Gelfand-Naimark theorem [16], which allows us to formulate the geometrical structure of Lie groups in terms of a Hopf structure [17].

In our previous work [18–26] we dealt with q-deformed versions of Minkowski space and Euclidean spaces. Their symmetries are described by q-analogs of the Lorentz group and the rotation group [27–31]. In addition to this, each q-deformed quantum space can be equipped with two different differential calculi [32–35]. These differential calculi can be combined with quantum groups to give q-deformations of Euclidean and Poincaré symmetry [36].

In Ref. [37] it was our aim to give a q-deformed version of analysis to quantum spaces of physical interest, i.e. Manin plane, q-deformed Euclidean space in three or four dimensions, and q-deformed Minkowski space. These considerations were mainly based on the ideas exposed in Refs. [38–44]. (The

reasonings in Refs. [45] and [46] go into the same direction.) In this manner we obtained a multi-dimensional version of the well-known q-calculus [47–49]. Using these results and following Refs. [50, 51] we introduced in Part I of this paper Fourier transformations and sesquilinear forms on q-deformed quantum spaces. The aim of Part II of this paper is to apply the new tools to describe wave functions on q-deformed position and momentum spaces.

To make Part II of the paper self-contained so that it can be read independently of Part I we first recall those results of Part I that are relevant in what follows. This task will be done in Sec. 2. In analogy to the undeformed case Fourier transformations on quantum spaces allow us to expand functions in terms of q-deformed exponentials and the q-deformed Fourier transform of a constant function leads us to q-analogs of delta functions. In Sec. 3 we identify q-deformed exponentials and q-deformed delta functions as eigenfunctions of momentum and position operators, respectively. For both sets of eigenfunctions we write down completeness and orthonormality relations. Combining our reasonings about Fourier transformations with those about sesquilinear forms we are able to calculate matrix elements of position and momentum operators in a basis of position or momentum eigenfunctions. For the sake of completeness a q-deformed version of the spectral decomposition for multiplication operators is formulated. With this result at hand we are in a position to propose q-analogs of multi-dimensional theta functions.

To give the formalism a physical meaning it is helpful to introduce the concept of quasipoints [50]. It should be mentioned that a quasipoint can be viewed as generalization of the idea of an eigenstate. In Sec. 5 we apply this concept to interpret our results from a physical point of view. In this respect we define q-analogs of expectation values and derive expressions for the probability to find a system in a certain quasipoint. Finally, Sec. 6 shows how to adapt our considerations to carry over to antisymmetrized quantum spaces. This way we obtain a q-deformed version of quantum kinematics with Grassmann variables. Sec. 7 closes our considerations by a short conclusion. For reference and for the purpose of introducing consistent and convenient notation, we provide a review of key notation and results in Appendix A.

2 Preliminaries

In this section, we collect definitions and relations from Part I that will be needed throughout Part II. First of all, let us recall the definition of

q-deformed Fourier transformations, which are given by

$$\begin{aligned}
\mathcal{F}_L(f)(p^k) &\equiv \int_{-\infty}^{+\infty} d_L^n x\, f(x^i) \overset{x}{\circledast} \exp(x^j|\mathrm{i}^{-1}p^k)_{\bar{R},L},\\
\mathcal{F}_{\bar{L}}(f)(p^k) &\equiv \int_{-\infty}^{+\infty} d_{\bar{L}}^n x\, f(x^i) \overset{x}{\circledast} \exp(x^j|\mathrm{i}^{-1}p^k)_{R,\bar{L}},
\end{aligned} \tag{1}$$

$$\begin{aligned}
\mathcal{F}_R(f)(p^k) &\equiv \int_{-\infty}^{+\infty} d_R^n x\, \exp(\mathrm{i}^{-1}p^k|x^j)_{R,\bar{L}} \overset{x}{\circledast} f(x^i),\\
\mathcal{F}_{\bar{R}}(f)(p^k) &\equiv \int_{-\infty}^{+\infty} d_{\bar{R}}^n x\, \exp(\mathrm{i}^{-1}p^k|x^j)_{\bar{R},L} \overset{x}{\circledast} f(x^i),
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
\mathcal{F}_L^*(f)(x^k) &\equiv \frac{1}{\mathrm{vol}_L} \int_{-\infty}^{+\infty} d_L^n p\, \exp(\mathrm{i}^{-1}p^l|\ominus_L x^k)_{\bar{R},L} \overset{x|p}{\odot}_{\bar{L}} f(p^j),\\
\mathcal{F}_{\bar{L}}^*(f)(x^k) &\equiv \frac{1}{\mathrm{vol}_{\bar{L}}} \int_{-\infty}^{+\infty} d_{\bar{L}}^n p\, \exp(\mathrm{i}^{-1}p^l|\ominus_{\bar{L}} x^k)_{R,\bar{L}} \overset{x|p}{\odot}_{L} f(\hat{p}^j),
\end{aligned} \tag{3}$$

$$\begin{aligned}
\mathcal{F}_R^*(f)(x^k) &\equiv \frac{1}{\mathrm{vol}_R} \int_{-\infty}^{+\infty} d_R^n p\, f(p^j) \overset{p|x}{\odot}_{\bar{R}} \exp(\ominus_R x^k|\mathrm{i}^{-1}p^l)_{R,\bar{L}},\\
\mathcal{F}_{\bar{R}}^*(f)(x^k) &\equiv \frac{1}{\mathrm{vol}_{\bar{R}}} \int_{-\infty}^{+\infty} d_{\bar{R}}^n p\, f(p^j) \overset{p|x}{\odot}_{R} \exp(\ominus_{\bar{R}} x^k|\mathrm{i}^{-1}p^l)_{\bar{R},L}.
\end{aligned} \tag{4}$$

The two types of Fourier transformations are inverse to each other in the sense that

$$\begin{aligned}
(\mathcal{F}_{\bar{R}}^* \circ \mathcal{F}_L)(f)(x^k) &= f(\kappa x^k),\\
(\mathcal{F}_R^* \circ \mathcal{F}_{\bar{L}})(f)(x^k) &= f(\kappa^{-1} x^k),
\end{aligned} \tag{5}$$

$$\begin{aligned}
(\mathcal{F}_{\bar{L}}^* \circ \mathcal{F}_R)(f)(x^k) &= f(\kappa^{-1} x^k),\\
(\mathcal{F}_L^* \circ \mathcal{F}_{\bar{R}})(f)(x^k) &= f(\kappa x^k),
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
(\mathcal{F}_L \circ \mathcal{F}_{\bar{R}}^*)(f)(x^k) &= \kappa^{-n} f(\kappa^{-1} x^k),\\
(\mathcal{F}_{\bar{L}} \circ \mathcal{F}_R^*)(f)(x^k) &= \kappa^{n} f(\kappa x^k),
\end{aligned} \tag{7}$$

$$(\mathcal{F}_R \circ \mathcal{F}_{\bar{L}}^*)(f)(x^k) = \kappa^{n} f(\kappa x^k),$$

$$(\mathcal{F}_{\bar{R}} \circ \mathcal{F}^*_L)(f)(x^k) = \kappa^{-n} f(\kappa^{-1} x^k), \tag{8}$$

where the values for κ are determined as follows:

(i) (quantum plane)

$$\kappa = \kappa_{\bar{L}} = \kappa_R = (\kappa_L)^{-1} = (\kappa_{\bar{R}})^{-1} = q^3, \tag{9}$$

(ii) (three-dimensional q-deformed Euclidean space)

$$\kappa = \kappa_{\bar{L}} = \kappa_R = (\kappa_L)^{-1} = (\kappa_{\bar{R}})^{-1} = q^6, \tag{10}$$

(iii) (four-dimensional q-deformed Euclidean space)

$$\kappa = \kappa_{\bar{L}} = \kappa_R = (\kappa_L)^{-1} = (\kappa_{\bar{R}})^{-1} = q^4, \tag{11}$$

(iv) (q-deformed Minkowski space)

$$\kappa = \kappa_{\bar{L}} = \kappa_R = (\kappa_L)^{-1} = (\kappa_{\bar{R}})^{-1} = q^{-4}. \tag{12}$$

The Fourier transformations in (1) and (2) can be used to introduce q-analogs of delta functions:

$$\begin{aligned} \delta^n_L(p^k) &\equiv \mathcal{F}_L(1)(p^k) = \int_{-\infty}^{+\infty} d^n_L x \exp(x^j|\mathrm{i}^{-1}p^k)_{\bar{R},L}, \\ \delta^n_{\bar{L}}(p^k) &\equiv \mathcal{F}_{\bar{L}}(1)(p^k) = \int_{-\infty}^{+\infty} d^n_{\bar{L}} x \exp(x^j|\mathrm{i}^{-1}p^k)_{R,\bar{L}}, \end{aligned} \tag{13}$$

$$\begin{aligned} \delta^n_R(p^k) &\equiv \mathcal{F}_R(1)(p^k) = \int_{-\infty}^{+\infty} d^n_R x \ \exp(\mathrm{i}^{-1}p^k|x^j)_{R,\bar{L}}, \\ \delta^n_{\bar{R}}(p^k) &\equiv \mathcal{F}_{\bar{R}}(1)(p^k) = \int_{-\infty}^{+\infty} d^n_{\bar{R}} x \ \exp(\mathrm{i}^{-1}p^k|x^j)_{\bar{R},L}. \end{aligned} \tag{14}$$

For the sake of brevity, integrals of delta functions get a name of their own:

$$\begin{aligned} \mathrm{vol}_L &\equiv \int_{-\infty}^{+\infty} d^n_{\bar{R}} p \, \delta^n_L(p^k) = \int_{-\infty}^{+\infty} d^n_L x \int_{-\infty}^{+\infty} d^n_{\bar{R}} p \exp(x^j|\mathrm{i}^{-1}p^k)_{\bar{R},L}, \\ \mathrm{vol}_{\bar{L}} &\equiv \int_{-\infty}^{+\infty} d^n_R p \, \delta^n_{\bar{L}}(p^k) = \int_{-\infty}^{+\infty} d^n_{\bar{L}} x \int_{-\infty}^{+\infty} d^n_R p \exp(x^j|\mathrm{i}^{-1}p^k)_{R,\bar{L}}, \end{aligned} \tag{15}$$

$$\mathrm{vol}_R \equiv \int_{-\infty}^{+\infty} d_{\bar{L}}^n p\, \delta_R^n(p^k) = \int_{-\infty}^{+\infty} d_{\bar{L}}^n p \int_{-\infty}^{+\infty} d_R^n x \exp(\mathrm{i}^{-1} p^k | x^j)_{R,\bar{L}},$$
$$\mathrm{vol}_{\bar{R}} \equiv \int_{-\infty}^{+\infty} d_L^n p\, \delta_{\bar{R}}^n(p^k) = \int_{-\infty}^{+\infty} d_L^n p \int_{-\infty}^{+\infty} d_{\bar{R}}^n x \exp(\mathrm{i}^{-1} p^k | x^j)_{\bar{R},L}, \tag{16}$$

where

$$\mathrm{vol}_L = \mathrm{vol}_{\bar{R}} \quad \text{and} \quad \mathrm{vol}_{\bar{L}} = \mathrm{vol}_R. \tag{17}$$

In analogy to ordinary delta functions q-deformed delta functions fulfill

$$\int_{-\infty}^{+\infty} d_A^n y\, f(y^i) \overset{y}{\circledast} \delta_B^n(y^j \oplus_C (\ominus_C\, x^k)) = \mathrm{vol}_{A,B}\, f(\kappa_C^{-1} x^k), \tag{18}$$

and

$$\int_{-\infty}^{+\infty} d_A^n y\, \delta_B^n((\ominus_C\, x^k) \oplus_C y^j) \overset{y}{\circledast} f(y^i) = \mathrm{vol}_{A,B}\, f(\kappa_C^{-1} x^k), \tag{19}$$

where

$$\mathrm{vol}_{A,B} \equiv \int_{-\infty}^{+\infty} d_A^n x\, \delta_B^n(x^k). \qquad A, B \in \{L, \bar{L}, R, \bar{R}\}. \tag{20}$$

As fundamental properties of the q-deformed Fourier transformations in (1) and (2) we have

$$\mathcal{F}_L(f \overset{x}{\triangleleft} \partial^j)(p^k) = \mathcal{F}_L(f)(p^k) \overset{p}{\circledast} (\mathrm{i}^{-1} p^j),$$
$$\mathcal{F}_{\bar{L}}(f \overset{x}{\bar{\triangleleft}} \hat{\partial}^j)(p^k) = \mathcal{F}_{\bar{L}}(f)(p^k) \overset{p}{\circledast} (\mathrm{i}^{-1} p^j), \tag{21}$$

$$\mathcal{F}_R(\hat{\partial}^j \overset{x}{\triangleright} f)(\hat{p}^k) = \mathrm{i}^{-1} p^j \overset{p}{\circledast} \mathcal{F}_R(f)(p^k),$$
$$\mathcal{F}_{\bar{R}}(\partial^j \overset{x}{\bar{\triangleright}} f)(p^k) = \mathrm{i}^{-1} p^j \overset{p}{\circledast} \mathcal{F}_{\bar{R}}(f)(p^k), \tag{22}$$

and

$$\mathcal{F}_L(f \overset{x}{\circledast} x^j)(p^k) = \mathrm{i}\mathcal{F}_L(f)(p^k) \overset{p}{\bar{\triangleleft}} \partial^j,$$
$$\mathcal{F}_{\bar{L}}(f \overset{x}{\circledast} x^j)(p^k) = \mathrm{i}\mathcal{F}_{\bar{L}}(f)(p^k) \overset{p}{\triangleleft} \hat{\partial}^j, \tag{23}$$

$$\mathcal{F}_R(x^j \overset{x}{\circledast} f)(p^k) = \mathrm{i}\hat{\partial}^j \overset{p}{\bar{\triangleright}} \mathcal{F}_R(f)(p^k),$$
$$\mathcal{F}_{\bar{R}}(x^j \overset{x}{\circledast} f)(p^k) = \mathrm{i}\partial^j \overset{p}{\triangleright} \mathcal{F}_{\bar{R}}(f)(p^k). \tag{24}$$

Similarly, the Fourier transformations in (3) and (4) are subject to

$$
\begin{aligned}
\mathcal{F}^*_L(\mathrm{i}\partial^j \overset{p}{\triangleright} f)(x^k) &= \kappa x^j \overset{x}{\circledast} \mathcal{F}^*_L(f)(x^k),\\
\mathcal{F}^*_{\bar{L}}(\mathrm{i}\hat{\partial}^j \overset{p}{\triangleright} f)(x^k) &= \kappa^{-1} x^j \overset{x}{\circledast} \mathcal{F}^*_{\bar{L}}(f)(x^k),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\mathcal{F}^*_R(\mathrm{i}f \overset{p}{\triangleleft} \hat{\partial}^j)(x^k) &= \kappa^{-1} \mathcal{F}^*_R(f)(x^k) \overset{x}{\circledast} x^j,\\
\mathcal{F}^*_{\bar{R}}(\mathrm{i}f \overset{p}{\bar{\triangleleft}} \partial^j)(x^k) &= \kappa \mathcal{F}^*_{\bar{R}}(f)(x^k) \overset{x}{\circledast} x^j,
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\mathcal{F}^*_L(\mathrm{i}^{-1}p^j \overset{p}{\circledast} f)(x^k) &= \kappa^{-1} \partial^j \overset{x}{\bar{\triangleright}} \mathcal{F}^*_L(f)(x^k),\\
\mathcal{F}^*_{\bar{L}}(\mathrm{i}^{-1}p^j \overset{p}{\circledast} f)(x^k) &= \kappa \hat{\partial}^j \overset{x}{\triangleright} \mathcal{F}^*_{\bar{L}}(f)(x^k),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\mathcal{F}^*_R(f \overset{p}{\circledast} (\mathrm{i}^{-1}p^j))(x^k) &= \kappa \mathcal{F}^*_R(f)(x^k) \overset{x}{\bar{\triangleleft}} \hat{\partial}^j,\\
\mathcal{F}^*_{\bar{R}}(f \overset{p}{\circledast} (\mathrm{i}^{-1}p^j))(x^k) &= \kappa^{-1} \mathcal{F}^*_{\bar{R}}(f)(x^k) \overset{x}{\triangleleft} \partial^j.
\end{aligned}
\tag{28}
$$

Our examinations need Fourier transforms of q-exponentials and q-delta functions. In the case of q-exponentials we find

$$
\begin{aligned}
&\mathcal{F}_L(\exp(\mathrm{i}^{-1}p^k|\ominus_L y^j)_{\bar{R},L})(x^i)\\
&\quad= \int_{-\infty}^{+\infty} d^n_L p\, \exp(\mathrm{i}^{-1}p^k|\ominus_L y^j)_{\bar{R},L} \overset{y|p}{\odot}_{\bar{L}} \exp(\mathrm{i}^{-1}p^l|x^i)_{\bar{R},L}\\
&\quad= \delta^n_L((\ominus_L y^j) \oplus_L x^i),
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\mathcal{F}_R(\exp(\ominus_{\bar{R}} y^j|\mathrm{i}^{-1}p^k)_{R,\bar{L}})(x^i)\\
&\quad= \int_{-\infty}^{+\infty} d^n_R p\, \exp(x^i|\mathrm{i}^{-1}p^l)_{R,\bar{L}} \overset{p|y}{\odot}_R \exp(\ominus_{\bar{R}} y^j|\mathrm{i}^{-1}p^k)_{R,\bar{L}}\\
&\quad= \delta^n_R(x^i \oplus_R (\ominus_R y^j)).
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
&\mathcal{F}^*_{\bar{R}}(\exp(\mathrm{i}\, y^j|p^k)_{\bar{R},L})(x^l)\\
&\quad= \frac{1}{\mathrm{vol}_{\bar{R}}} \int_{-\infty}^{+\infty} d^n_{\bar{R}} p\, \exp(\mathrm{i}\, y^j|p^k)_{\bar{R},L} \overset{p|x}{\odot}_R \exp(\mathrm{i}\, x^l|\ominus_L p^l)_{\bar{R},L}
\end{aligned}
$$

$$= \frac{1}{\mathrm{vol}_{\bar{R}}}\,\delta^n_{\bar{R}}(y^j \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^l)), \tag{31}$$

$$\begin{aligned}
&\mathcal{F}^*_{\bar{L}}(\exp(\mathrm{i}^{-1}p^k|y^j)_{R,\bar{L}})(x^l)\\
&\quad = \frac{1}{\mathrm{vol}_{\bar{L}}}\int_{-\infty}^{+\infty} d^n_{\bar{L}}p\,\exp(\mathrm{i}^{-1}p^m|\ominus_{\bar{L}}\, x^l)_{R,\bar{L}} \overset{x|p}{\odot}_L \exp(\mathrm{i}^{-1}p^k y^j)_{R,\bar{L}}\\
&\quad = \frac{1}{\mathrm{vol}_{\bar{L}}}\,\delta^n_{\bar{L}}((\ominus_{\bar{L}}\, x^l) \oplus_{\bar{L}} y^j).
\end{aligned} \tag{32}$$

The Fourier transforms of q-delta functions take the form

$$\begin{aligned}
&\mathcal{F}_L(\delta^n_{\bar{R}}(y^j \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^i)))(p^k)\\
&\quad = \int_{-\infty}^{+\infty} d^n_L x\,\delta^n_{\bar{R}}(y^j \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^i)) \overset{x}{\circledast} \exp(x^l|\mathrm{i}^{-1}p^k)_{\bar{R},L}\\
&\quad = \kappa^{-n}\mathrm{vol}_{\bar{R}}\exp(\kappa^{-1}y^j|\mathrm{i}^{-1}p^k)_{\bar{R},L},
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\mathcal{F}_R(\delta^n_{\bar{L}}((\ominus_{\bar{L}}\, x^i) \oplus_{\bar{L}} y^j))(p^k)\\
&\quad = \int_{-\infty}^{+\infty} d^n_R x\,\exp(\mathrm{i}^{-1}p^k|x^l)_{R,\bar{L}} \overset{x}{\circledast} \delta^n_{\bar{L}}((\ominus_{\bar{L}}\, x^i) \oplus_{\bar{L}} y^j)\\
&\quad = \kappa^n\mathrm{vol}_{\bar{L}}\exp(\mathrm{i}^{-1}p^k|\kappa y^j)_{R,\bar{L}},
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
&\mathcal{F}^*_{\bar{R}}(\delta^n_L((\ominus_L\, y^j) \oplus_L x^i))(p^k)\\
&\quad = \frac{1}{\mathrm{vol}_{\bar{R}}}\int_{-\infty}^{+\infty} d^n_{\bar{R}}x\,\delta^n_L((\ominus_L\, y^j) \oplus_L x^i) \overset{yx|p}{\odot}_R \exp(\mathrm{i}^{-1}p^k|\ominus_L\, x^l)_{\bar{R},L}\\
&\quad = \exp(\mathrm{i}^{-1}p^k|\ominus_L\, y^j)_{\bar{R},L},
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\mathcal{F}_{\bar{L}}(\delta^n_R(x^i \oplus_R (\ominus_R\, y^j)))(p^k)\\
&\quad = \frac{1}{\mathrm{vol}_{\bar{L}}}\int_{-\infty}^{+\infty} d^n_{\bar{L}}x\,\exp(\ominus_R\, x^l|\mathrm{i}^{-1}\,p^k)_{R,\bar{L}} \overset{p|xy}{\odot}_L \delta^n_R(x^i \oplus_R (\ominus_R\, y^j))\\
&\quad = \exp(\ominus_R\, y^j|\mathrm{i}^{-1}p^k)_{R,\bar{L}}.
\end{aligned} \tag{36}$$

The corresponding relations for the other versions of q-deformed Fourier transforms are obtained most easily from the above formulae by applying

the substitutions

$$L \to \bar{L}, \quad \bar{L} \to L, \quad R \to \bar{R}, \quad \bar{R} \to R, \quad \kappa \to \kappa^{-1}. \tag{37}$$

Next, we come to sesquilinear forms on quantum spaces. Obviously, they should be given by

$$\begin{aligned}
\big\langle f, g \big\rangle_A &\equiv \int_{-\infty}^{+\infty} d_A^n x \, \overline{f(x^i)} \overset{x}{\circledast} g(x^j), \\
\big\langle f, g \big\rangle'_A &\equiv \int_{-\infty}^{+\infty} d_A^n x \, f(x^i) \overset{x}{\circledast} \overline{g(x^j)},
\end{aligned} \tag{38}$$

where $A \in \{L, \bar{L}, R, \bar{R}\}$. However, the conjugation properties of q-deformed integrals are responsible for the fact that these sesquilinear forms are not symmetrical. To circumvent this problem one can take the expressions

$$\begin{aligned}
\big\langle f, g \big\rangle_1 &\equiv \frac{\mathrm{i}^n}{2} \big(\big\langle f, g \big\rangle_L + \big\langle f, g \big\rangle_{\bar{R}} \big), \\
\big\langle f, g \big\rangle_2 &\equiv \frac{\mathrm{i}^n}{2} \big(\big\langle f, g \big\rangle_{\bar{L}} + \big\langle f, g \big\rangle_R \big),
\end{aligned} \tag{39}$$

$$\begin{aligned}
\big\langle f, g \big\rangle'_1 &\equiv \frac{\mathrm{i}^n}{2} \big(\big\langle f, g \big\rangle'_L + \big\langle f, g \big\rangle'_{\bar{R}} \big), \\
\big\langle f, g \big\rangle'_2 &\equiv \frac{\mathrm{i}^n}{2} \big(\big\langle f, g \big\rangle'_{\bar{L}} + \big\langle f, g \big\rangle'_R \big).
\end{aligned} \tag{40}$$

For later purpose let us mention that the sesquilinear forms in (38) can be used to introduce the adjoint of an operator. The adjoints of partial derivatives, for example, can be read off from the identities

$$\begin{aligned}
\big\langle f, \partial^i \triangleright g \big\rangle_A &= \big\langle \overline{\partial^i} \,\bar{\triangleright}\, f, g \big\rangle_A, \\
\big\langle f, \hat{\partial}^i \,\bar{\triangleright}\, g \big\rangle_A &= \big\langle \overline{\hat{\partial}^i} \triangleright f, g \big\rangle_A,
\end{aligned} \tag{41}$$

$$\begin{aligned}
\big\langle f \triangleleft \hat{\partial}^i, g \big\rangle'_A &= \big\langle f, g \,\bar{\triangleleft}\, \overline{\hat{\partial}^i} \big\rangle'_A, \\
\big\langle f \,\bar{\triangleleft}\, \partial^i, g \big\rangle'_A &= \big\langle f, g \triangleleft \overline{\partial^i} \big\rangle'_A.
\end{aligned} \tag{42}$$

Essentially for us is the fact that our sesquilinear forms obey q-analogs of the Fourier-Plancherel identity. In Part I of this article it was shown that

$$\big\langle f, g \big\rangle'_{L,x} = (-1)^n \big\langle \mathcal{F}_L(f), \mathcal{F}^*_{\bar{R}}(g)(\kappa^{-1} p^i) \big\rangle'_{\bar{R},p},$$

$$\big\langle f,g\big\rangle'_{\bar{L},x} = (-1)^n \big\langle \mathcal{F}_{\bar{L}}(f), \mathcal{F}^*_R(g)(\kappa p^i)\big\rangle'_{R,p}, \tag{43}$$

$$\begin{aligned}\big\langle f,g\big\rangle'_{R,x} &= (-1)^n \big\langle \mathcal{F}^*_R(f)(\kappa p^i), \mathcal{F}_{\bar{L}}(g)\big\rangle'_{\bar{L},p},\\ \big\langle f,g\big\rangle'_{\bar{R},x} &= (-1)^n \big\langle \mathcal{F}^*_{\bar{R}}(f)(\kappa^{-1} p^i), \mathcal{F}_L(g)\big\rangle'_{L,p},\end{aligned} \tag{44}$$

and

$$\begin{aligned}\big\langle f,g\big\rangle_{L,x} &= (-1)^n \big\langle \mathcal{F}_{\bar{R}}(f), \mathcal{F}^*_L(g)(\kappa^{-1} p^i)\big\rangle_{\bar{R},p},\\ \big\langle f,g\big\rangle_{\bar{L},x} &= (-1)^n \big\langle \mathcal{F}_R(f), \mathcal{F}^*_{\bar{L}}(g)(\kappa p^i)\big\rangle_{R,p},\end{aligned} \tag{45}$$

$$\begin{aligned}\big\langle f,g\big\rangle_{R,x} &= (-1)^n \big\langle \mathcal{F}^*_{\bar{L}}(f)(\kappa p^i), \mathcal{F}_R(g)\big\rangle_{\bar{L},p},\\ \big\langle f,g\big\rangle_{\bar{R},x} &= (-1)^n \big\langle \mathcal{F}^*_L(f)(\kappa^{-1} p^i), \mathcal{F}_{\bar{R}}(g)\big\rangle_{L,p}.\end{aligned} \tag{46}$$

To avoid the additional minus signs in the above formulae, we modify the definitions of Fourier transformations, delta functions, and volume elements by carrying out the following substitutions in the defining expressions:

$$\begin{aligned}\int d^n_L x, \int d^n_{\bar{R}} x &\longrightarrow \int d^n_1 x \equiv \frac{\mathrm{i}}{2}\Big(\int d^n_L x + \int d^n_{\bar{R}} x\Big),\\ \int d^n_{\bar{L}} x, \int d^n_R x &\longrightarrow \int d^n_2 x \equiv \frac{\mathrm{i}}{2}\Big(\int d^n_{\bar{L}} x + \int d^n_R x\Big).\end{aligned} \tag{47}$$

That means we deal with real integrals, only. This way, we obtain new objects which are distinguished from the original ones by a tilde:

$$\begin{aligned}&\mathcal{F}_A \longrightarrow \tilde{\mathcal{F}}_A, \quad \mathcal{F}^*_A \longrightarrow \tilde{\mathcal{F}}^*_A,\\ &\delta^n_A \longrightarrow \tilde{\delta}^n_A, \quad \mathrm{vol}_A \longrightarrow \widetilde{\mathrm{vol}}_A.\end{aligned} \tag{48}$$

It should be mentioned that all identities presented so far remain unchanged under these substitutions. However, there is one exception, since the Fourier-Plancherel identities now become

$$\begin{aligned}\big\langle f,g\big\rangle'_{1,x} &= \big\langle \tilde{\mathcal{F}}_L(f), \tilde{\mathcal{F}}^*_{\bar{R}}(g)(\kappa^{-1}p^i)\big\rangle'_{1,p} = \big\langle \tilde{\mathcal{F}}^*_{\bar{R}}(f)(\kappa^{-1}p^i), \tilde{\mathcal{F}}_L(g)\big\rangle'_{1,p},\\ \big\langle f,g\big\rangle'_{2,x} &= \big\langle \tilde{\mathcal{F}}_{\bar{L}}(f), \tilde{\mathcal{F}}^*_R(g)(\kappa p^i)\big\rangle'_{2,p} = \big\langle \tilde{\mathcal{F}}^*_R(f)(\kappa p^i), \tilde{\mathcal{F}}_{\bar{L}}(g)\big\rangle'_{2,p},\end{aligned} \tag{49}$$

and

$$\begin{aligned}\big\langle f,g\big\rangle_{1,x} &= \big\langle \tilde{\mathcal{F}}_{\bar{R}}(f), \tilde{\mathcal{F}}^*_L(g)(\kappa^{-1}p^i)\big\rangle_{1,p} = \big\langle \tilde{\mathcal{F}}^*_L(f)(\kappa^{-1}p^i), \tilde{\mathcal{F}}_{\bar{R}}(g)\big\rangle_{1,p},\\ \big\langle f,g\big\rangle_{2,x} &= \big\langle \tilde{\mathcal{F}}_R(f), \tilde{\mathcal{F}}^*_{\bar{L}}(g)(\kappa p^i)\big\rangle_{2,p} = \big\langle \tilde{\mathcal{F}}^*_{\bar{L}}(f)(\kappa p^i), \tilde{\mathcal{F}}_R(g)\big\rangle_{2,p}.\end{aligned} \tag{50}$$

3 Wave functions and matrix representations

In this section we introduce q-analogs of position and momentum eigenfunctions and present a systematic study of their properties, i.e. we show that both sets of eigenfunctions are orthonormal and complete. In addition to this we calculate the matrix elements of position and momentum operators in a position basis as well as a momentum basis. Finally, we examine how the idea of a spectrum fits into our considerations. This way, we lay the foundations for a physical interpretation of our formalism.

3.1 Definition of position and momentum space eigenfunctions

In analogy to the classical case a *momentum eigenfunction* has to satisfy

$$\mathrm{i}\partial^i \triangleright u_p(x^j) = u_p(x^j) \overset{p}{\circledast} p^i, \tag{51}$$

or alternatively

$$\bar{u}_p(x^j) \triangleleft (\mathrm{i}\partial^i) = p^i \overset{p}{\circledast} \bar{u}_p(x^j). \tag{52}$$

Functions with one of these properties are given by

$$\begin{aligned}(u_{\bar{R},L})_p(x^i) &\equiv (\mathrm{vol}_L)^{-1/2} \exp(x^i|\mathrm{i}^{-1}p^k)_{\bar{R},L}, \\ (u_{R,\bar{L}})_p(x^i) &\equiv (\mathrm{vol}_{\bar{L}})^{-1/2} \exp(x^i|\mathrm{i}^{-1}p^k)_{R,\bar{L}},\end{aligned} \tag{53}$$

$$\begin{aligned}(\bar{u}_{\bar{R},L})_p(x^i) &\equiv (\mathrm{vol}_{\bar{R}})^{-1/2} \exp(\mathrm{i}^{-1}p^k|x^i)_{\bar{R},L}, \\ (\bar{u}_{R,\bar{L}})_p(x^i) &\equiv (\mathrm{vol}_R)^{-1/2} \exp(\mathrm{i}^{-1}p^k|x^i)_{R,\bar{L}}.\end{aligned} \tag{54}$$

For a *position eigenfunction* we require to hold that

$$x^i \overset{x}{\circledast} u_y(x^j) = u_y(x^j) \overset{y}{\circledast} y^i, \tag{55}$$

or

$$\bar{u}_y(x^j) \overset{x}{\circledast} x^i = y^i \overset{y}{\circledast} \bar{u}_y(x^j). \tag{56}$$

We can choose

$$\begin{aligned}(u_A)_y(x^i) &\equiv (\mathrm{vol}_A)^{-1}\, \delta^n_A(x^i \oplus_A (\ominus_A \kappa_A y^j)) \\ &= (\mathrm{vol}_A)^{-1}\, \delta^n_A((\ominus_A \kappa_A x^i) \oplus_A y^j),\end{aligned} \tag{57}$$

and

$$
\begin{aligned}
(\bar{u}_A)_y(x^i) &\equiv (\mathrm{vol}_A)^{-1}\, \delta^n_A((\ominus_A y^j) \oplus_A x^i) \\
&= (\mathrm{vol}_A)^{-1}\, \delta^n_A(y^j \oplus_A (\ominus_A \kappa_A x^i)),
\end{aligned} \tag{58}
$$

where $A \in \{L, \bar{L}, R, \bar{R}\}$. That these functions are indeed position eigenfunctions can be seen by the following consideration. From the identities for q-deformed delta functions [cf. Eqs. (18) and (19)] we know that

$$
\begin{aligned}
\int dx^n_{\bar{R}}\, (\bar{u}_L)_y(x^j) \overset{x}{\circledast} x^i &= \int_{-\infty}^{+\infty} dx^n_{\bar{R}}\, (\mathrm{vol}_L)^{-1}\, \delta^n_L((\ominus_L\, \kappa^{-1} y^k) \oplus_L x^j) \overset{x}{\circledast} x^i \\
&= y^i = \int_{-\infty}^{+\infty} dx^n_{\bar{R}}\, (\mathrm{vol}_L)^{-1/2}\, y^i\, \delta^n_L(x^j) \\
&= \int_{-\infty}^{+\infty} dx^n_{\bar{R}}\, (\mathrm{vol}_L)^{-1}\, y^i \overset{y}{\circledast} \delta^n_L((\ominus_L\, \kappa^{-1} y^k) \oplus_L x^j) \\
&= \int_{-\infty}^{+\infty} dx^n_{\bar{R}}\, y^i \overset{y}{\circledast} (\bar{u}_L)_y(x^j),
\end{aligned} \tag{59}
$$

where the fourth equality results from translation invariance of integrals over the whole space. Thus, we have shown that

$$
\int_{-\infty}^{+\infty} dx^n_{\bar{R}} \left((\bar{u}_L)_y(x^j) \overset{x}{\circledast} x^i - y^i \overset{y}{\circledast} (\bar{u}_L)_y(x^j) \right) = 0, \tag{60}
$$

which implies

$$
\begin{aligned}
(\bar{u}_L)_y(x^j) \overset{x}{\circledast} x^i - y^i \overset{y}{\circledast} (\bar{u}_L)_y(x^j) &= \varphi(x^i), \\
d\varphi(x^i) &= 0.
\end{aligned} \tag{61}
$$

From the discussion in Ref. [37] we know that $\varphi(x^i)$ has to vanish on a q-lattice and this observation should allow us to assume $\varphi(x^i) = 0$. Similar arguments apply for the other position eigenfunctions.

Before proceeding any further let us be a little bit more precise. The relations (51) and (52) characterize the eigenfunctions of momentum and position operators in position space. We can also ask for eigenfunctions of momentum operators in momentum space. In our approach position and momentum variables play symmetrical roles. Thus, eigenfunctions of momentum operators in momentum space are obtained from the functions in (57) and (58) by replacing position variables with momentum variables. A

short glance at the identities in (29)-(32) tells us that q-deformed Fourier transformations map the momentum eigenfunctions in position space to those in momentum space and vice versa. This means that q-deformed Fourier transformations organize a change of basis. To be more specific, we have

$$\mathcal{F}_L((\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}}p}(y^i))(x^j) = (\mathrm{vol}_L)^{1/2}(\bar{u}_L)_{\kappa y}(x^j),$$
$$\mathcal{F}_R((u_{R,\bar{L}})_{\ominus_{\bar{L}}p}(y^i))(x^j) = (\mathrm{vol}_R)^{1/2}(u_R)_{\kappa^{-1}y}(x^j), \tag{62}$$

and

$$\mathcal{F}^*_{\bar{R}}((u_{\bar{R},L})_p(y^i))(x^j) = (\mathrm{vol}_{\bar{R}})^{1/2}(\bar{u}_{\bar{R}})_y(\kappa^{-1}x^j),$$
$$\mathcal{F}^*_{\bar{L}}((\bar{u}_{R,\bar{L}})_p(y^i))(x^j) = (\mathrm{vol}_{\bar{L}})^{1/2}(u_{\bar{L}})_y(\kappa x^j). \tag{63}$$

3.2 Completeness of position and momentum eigenfunctions

Now, we are ready to prove that each wave function can be expanded in terms of position eigenfunctions as well as momentum eigenfunctions. In what follows we concentrate attention on two q-geometries, only, since the results for the other ones are obtained most easily by means of the substitutions

$$L \leftrightarrow \bar{L}, \quad R \leftrightarrow \bar{R}, \quad \triangleright \leftrightarrow \bar{\triangleright}, \quad \triangleleft \leftrightarrow \bar{\triangleleft}, \quad \partial \leftrightarrow \hat{\partial}, \quad \kappa \leftrightarrow \kappa^{-1}. \tag{64}$$

The vector representing a wave function $\psi(x^i)$ in momentum space is given by the Fourier transform of $\psi(x^i)$, i.e.

$$(c_L)_p = \mathcal{F}_L(\psi)(p^k) = \big\langle \psi, \exp(\mathrm{i}^{-1}p^k|x^j)_{\bar{R},L}\big\rangle'_{L,x}$$
$$= (\mathrm{vol}_{\bar{R}})^{1/2}\big\langle \psi, (\bar{u}_{\bar{R},L})_p(x^j)\big\rangle'_{L,x}, \tag{65}$$

$$(c_R)_p = \mathcal{F}_R(\psi)(p^k) = \big\langle \exp(x^j|\mathrm{i}^{-1}p^k)_{R,\bar{L}}, \psi\big\rangle_{R,x}$$
$$= (\mathrm{vol}_{\bar{L}})^{1/2}\big\langle (u_{R,\bar{L}})_p(x^j), \psi\big\rangle_{R,x}, \tag{66}$$

and

$$(c^*_L)_p = \mathcal{F}^*_L(\psi)(p^k) = (\mathrm{vol}_{\bar{R}})^{-1}\big\langle \exp(\ominus_{\bar{R}}\, x^j|\mathrm{i}^{-1}p^k)_{\bar{R},L}, \psi\big\rangle_{L,x}$$
$$= (\mathrm{vol}_{\bar{R}})^{-1/2}\big\langle (u_{\bar{R},L})_p(\ominus_{\bar{R}}\, x^j), \psi\big\rangle_{L,x}, \tag{67}$$

$$(c^*_R)_p = \mathcal{F}^*_R(\psi)(p^k) = (\mathrm{vol}_{\bar{L}})^{-1}\big\langle \psi, \exp(\mathrm{i}^{-1}p^k|\ominus_{\bar{L}}\, x^j)_{R,\bar{L}}\big\rangle'_{R,x}$$

$$= (\mathrm{vol}_{\bar{L}})^{-1/2} \big\langle \psi, (\bar{u}_{\bar{R},L})_p(\ominus_{\bar{L}}\, x^j) \big\rangle'_{R,x}. \tag{68}$$

Clearly, the above Fourier coefficients determine the expansions in terms of momentum eigenfunctions [cf. the identities in (5)-(8)], i.e.

$$\begin{aligned}
\psi(x^i) &= \mathcal{F}^*_{\bar{R}}(\mathcal{F}_L(\psi)(p^k))(\kappa^{-1}x^i) \\
&= \kappa^n (\mathrm{vol}_L)^{-1/2} \int_{-\infty}^{+\infty} d^n_{\bar{R}}p\, (c_L)_{\kappa p} \overset{p|x}{\odot}_R (u_{\bar{R},L})_p(\ominus_{\bar{R}}\, x^i) \\
&= \kappa^n (\mathrm{vol}_L)^{-1/2} \big\langle (c_L)_{\kappa p}, (\bar{u}_{\bar{R},L})_p(\ominus_L\, x^i) \big\rangle'_{\bar{R},p},
\end{aligned} \tag{69}$$

$$\begin{aligned}
\psi(x^i) &= \mathcal{F}^*_{\bar{L}}(\mathcal{F}_R(\psi)(p^k))(\kappa x^i) \\
&= \kappa^{-n} (\mathrm{vol}_R)^{-1/2} \int_{-\infty}^{+\infty} d^n_{\bar{L}}p\, (\bar{u}_{R,\bar{L}})_p(\ominus_{\bar{L}}\, x^i) \overset{x|p}{\odot}_L (c_R)_{\kappa^{-1}p} \\
&= \kappa^{-n} (\mathrm{vol}_R)^{-1/2} \big\langle (u_{R,\bar{L}})_p(\ominus_R\, x^i), (c_R)_{\kappa^{-1}p} \big\rangle_{\bar{L},p},
\end{aligned} \tag{70}$$

and

$$\begin{aligned}
\psi(x^i) &= \kappa^n \mathcal{F}_{\bar{R}}(\mathcal{F}^*_L(\psi)(p^k))(\kappa x^i) \\
&= (\mathrm{vol}_{\bar{R}})^{1/2} \int_{-\infty}^{+\infty} d^n_{\bar{R}}p\, (u_{\bar{R},L})_p(x^i) \overset{p}{\circledast} (c^*_L)_{\kappa^{-1}p} \\
&= (\mathrm{vol}_{\bar{R}})^{1/2} \big\langle (\bar{u}_{\bar{R},L})_p(x^i), (c^*_L)_{\kappa^{-1}p} \big\rangle_{\bar{R},p},
\end{aligned} \tag{71}$$

$$\begin{aligned}
\psi(x^i) &= \kappa^{-n} \mathcal{F}_{\bar{L}}(\mathcal{F}^*_R(\psi)(p^k))(\kappa^{-1}x^i) \\
&= (\mathrm{vol}_{\bar{L}})^{1/2} \int_{-\infty}^{+\infty} d^n_{\bar{L}}p\, (c^*_R)_{\kappa p} \overset{p}{\circledast} (\bar{u}_{R,\bar{L}})_p(x^i) \\
&= (\mathrm{vol}_{\bar{L}})^{1/2} \big\langle (c^*_R)_{\kappa p}, (u_{R,\bar{L}})_p(x^i) \big\rangle'_{\bar{L},p}.
\end{aligned} \tag{72}$$

This way we see that the set of momentum eigenfunctions is complete and the corresponding completeness relations become

$$\begin{aligned}
&\big\langle (u_{\bar{R},L})_p(x^i), (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}}p}(y^j) \big\rangle'_{\bar{R},p} = \big\langle (\bar{u}_{\bar{R},L})_p(x^i), (u_{\bar{R},L})_{\ominus_L p}(y^j) \big\rangle_{\bar{R},p} \\
&\quad = \int_{-\infty}^{+\infty} d^n_{\bar{R}}p\, (u_{\bar{R},L})_p(x^i) \overset{p|y}{\odot}_R (u_{\bar{R},L})_{\ominus_L p}(y^j) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-1/2} \mathcal{F}_{\bar{R}}((u_{\bar{R},L})_{\ominus_L p}(y^j))(x^i) = (\mathrm{vol}_{\bar{R}})^{1/2} \mathcal{F}^*_{\bar{R}}((u_{\bar{R},L})_p(x^i))(y^j) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-1} \delta^n_{\bar{R}}(x^i \oplus_{\bar{R}} (\ominus_{\bar{R}}\, y^j)),
\end{aligned} \tag{73}$$

$$\big\langle (\bar{u}_{R,\bar{L}})_{\ominus_R p}(y^j), (u_{R,\bar{L}})_p(x^i) \big\rangle'_{\bar{L},p} = \big\langle (u_{R,\bar{L}})_{\ominus_{\bar{L}}p}(y^j), (\bar{u}_{R,\bar{L}})_p(x^i) \big\rangle_{\bar{L},p}$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} d_{\bar{L}}^n p\, (\bar{u}_{R,\bar{L}})_{\ominus_R p}(y^j) \overset{y|p}{\odot}_L (\bar{u}_{R,\bar{L}})_p(x^i) \\
&= (\mathrm{vol}_{\bar{L}})^{-1/2} \mathcal{F}_{\bar{L}}((\bar{u}_{R,\bar{L}})_{\ominus_R p}(y^j)(x^i) = (\mathrm{vol}_{\bar{L}})^{1/2} \mathcal{F}^*_{\bar{L}}((\bar{u}_{R,\bar{L}})_p(x^i))(y^j) \\
&= (\mathrm{vol}_{\bar{L}})^{-1} \delta_{\bar{L}}^n((\ominus_{\bar{L}}\, y^j) \oplus_{\bar{L}} x^i). \qquad (74)
\end{aligned}
$$

The correspondence between the above completeness relations and the expansions in (69)-(72) is illustrated by the following calculation:

$$
\begin{aligned}
\psi(x^i) &= \mathcal{F}^*_{\bar{R}}(\mathcal{F}_L(\psi))(\kappa^{-1}x^i) \\
&= \int_{-\infty}^{+\infty} d_{\bar{R}}^n p \int_{-\infty}^{+\infty} d_L^n y\, \psi(y^l) \overset{y}{\circledast} (u_{\bar{R},L})_p(y^k) \overset{p|x}{\odot}_R (u_{\bar{R},L})_p(\ominus_{\bar{R}}\, \kappa^{-1}x^i) \\
&= \int_{-\infty}^{+\infty} d_L^n y\, \psi(y^l) \overset{y}{\circledast} \int_{-\infty}^{+\infty} d_{\bar{R}}^n p\, (u_{\bar{R},L})_p(y^k) \overset{p|x}{\odot}_R (u_{\bar{R},L})_{\ominus_L p}(\kappa^{-1}x^i) \\
&= (\mathrm{vol}_{\bar{R}})^{-1} \int_{-\infty}^{+\infty} d_L^n y\, \psi(y^l) \overset{y}{\circledast} \delta_{\bar{R}}^n(y^k \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1}x^i)). \qquad (75)
\end{aligned}
$$

Likewise we have

$$
\begin{aligned}
\psi(x^i) &= \mathcal{F}^*_{\bar{L}}(\mathcal{F}_R(\psi))(\kappa x^i) \\
&= \int_{-\infty}^{+\infty} d_R^n y \int_{-\infty}^{+\infty} d_{\bar{L}}^n p\, (\bar{u}_{R,\bar{L}})_{\ominus_R p}(\kappa x^i) \overset{x|p}{\odot}_L (\bar{u}_{R,\bar{L}})_p(y^k) \overset{y}{\circledast} \psi(y^l) \\
&= (\mathrm{vol}_{\bar{L}})^{-1} \int_{-\infty}^{+\infty} d_R^n y\, \delta_{\bar{L}}^n((\ominus_{\bar{L}}\, \kappa x^i) \oplus_{\bar{L}} y^k) \overset{y}{\circledast} \psi(y^l), \qquad (76)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi(x^i) &= \kappa^{-n} \mathcal{F}_{\bar{L}}(\mathcal{F}^*_R(\psi))(\kappa^{-1}x^i) \\
&= \kappa^{-n} \int_{-\infty}^{+\infty} d_R^n y \int_{-\infty}^{+\infty} d_{\bar{L}}^n p\, \psi(y^l) \overset{y|p}{\odot}_{\bar{R}} (\bar{u}_{R,\bar{L}})_{\ominus_R p}(y^k) \\
&\qquad\qquad \overset{y|p}{\odot}_L (\bar{u}_{R,\bar{L}})_p(\kappa^{-1}x^i) \\
&= \kappa^{-n} (\mathrm{vol}_{\bar{L}})^{-1} \int_{-\infty}^{+\infty} d_R^n y\, \psi(y^l) \overset{y}{\circledast} \delta_{\bar{L}}^n((\ominus_{\bar{L}}\, y^k) \oplus_{\bar{L}} \kappa^{-1}x^i), \qquad (77)
\end{aligned}
$$

$$
\begin{aligned}
\psi(x^i) &= \kappa^n \mathcal{F}_{\bar{R}}(\mathcal{F}^*_L(\psi))(\kappa x^i) \\
&= \kappa^n \int_{-\infty}^{+\infty} d_L^n y \int_{-\infty}^{+\infty} d_{\bar{R}}^n p\, (u_{\bar{R},L})_p(\kappa x^i) \overset{p|y}{\odot}_R (u_{\bar{R},L})_{\ominus_L p}(y^k) \overset{p|y}{\odot}_{\bar{L}} \psi(y^l)
\end{aligned}
$$

$$= \kappa^n (\mathrm{vol}_{\bar{R}})^{-1} \int_{-\infty}^{+\infty} d_L^n y \, \delta_{\bar{R}}^n(\kappa x^i \oplus_{\bar{R}} (\ominus_{\bar{R}} \, y^k)) \overset{y}{\circledast} \psi(y^l). \tag{78}$$

Now, we turn to position eigenfunctions. The vector representing the wave function $\psi(x^i)$ in a basis of position eigenfunctions is given by the wave function itself. For the expansion coefficients we have

$$\begin{aligned}
(c_L)_y &= (-1)^n \big\langle (u_L)_y(x^j), \psi(x^i) \big\rangle_{L,x} \\
&= \int_{-\infty}^{+\infty} d_L^n x \, (\bar{u}_{\bar{R}})_y(x^k) \overset{x}{\circledast} \psi(x^i) \\
&= (\mathrm{vol}_{\bar{R}})^{-1} \int_{-\infty}^{+\infty} d_L^n x \, \delta_{\bar{R}}^n((\ominus_{\bar{R}} \, \kappa^{-1} y^j) \oplus_{\bar{R}} x^k) \overset{x}{\circledast} \psi(x^i) \\
&= \psi(y^j),
\end{aligned} \tag{79}$$

$$\begin{aligned}
(c_R)_y &= (-1)^n \big\langle (u_R)_y(x^j), \psi(x^i) \big\rangle_{R,x} \\
&= \int_{-\infty}^{+\infty} d_R^n x \, (\bar{u}_{\bar{L}})_y(x^k) \overset{x}{\circledast} \psi(x^i) \\
&= (\mathrm{vol}_{\bar{L}})^{-1} \int_{-\infty}^{+\infty} d_R^n x \, \delta_{\bar{L}}^n((\ominus_{\bar{L}} \, \kappa y^j) \oplus_{\bar{L}} x^k)) \overset{x}{\circledast} \psi(x^i) \\
&= \psi(y^j),
\end{aligned} \tag{80}$$

and

$$\begin{aligned}
(c'_L)_y &= (-1)^n \big\langle \psi(x^i), (\bar{u}_L)_y(x^j) \big\rangle'_{L,x} \\
&= \int_{-\infty}^{+\infty} d_L^n x \, \psi(x^i) \overset{x}{\circledast} (u_{\bar{R}})_y(x^k) \\
&= (\mathrm{vol}_{\bar{R}})^{-1} \int_{-\infty}^{+\infty} d_L^n x \, \psi(x^i) \overset{x}{\circledast} \delta_{\bar{R}}^n(x^k \oplus_{\bar{R}} (\ominus_{\bar{R}} \, \kappa^{-1} y^j)) \\
&= \psi(y^j),
\end{aligned} \tag{81}$$

$$\begin{aligned}
(c'_R)_y &= (-1)^n \big\langle \psi(x^i), (\bar{u}_R)_y(x^j) \big\rangle'_{R,x} \\
&= \int_{-\infty}^{+\infty} d_R^n x \, \psi(x^i) \overset{x}{\circledast} (u_{\bar{L}})_y(x^k) \\
&= (\mathrm{vol}_{\bar{L}})^{-1} \int_{-\infty}^{+\infty} d_R^n x \, \psi(x^i) \overset{x}{\circledast} \delta_{\bar{L}}^n(x^k \oplus_{\bar{L}} (\ominus_{\bar{L}} \, \kappa y^j)) \\
&= \psi(y^j).
\end{aligned} \tag{82}$$

The minus signs in each of the above calculations is due to the conjugation properties of q-deformed delta functions and the last step is an application of the identities in (18) and (19).

With the coefficients in (79)-(82) the expansions in terms of position eigenfunctions take the form

$$\begin{aligned}\psi(x^i) &= (-1)^n \big\langle (\bar{u}_L)_y(x^i), (c_L)_y \big\rangle_{L,y} \\ &= \int_{-\infty}^{+\infty} d_L^n y\, (u_{\bar{R}})_y(x^i) \overset{y}{\circledast} (c_L)_y = (c_L)_x, \end{aligned} \tag{83}$$

$$\begin{aligned}\psi(x^i) &= (-1)^n \big\langle (\bar{u}_R)_y(x^i), (c_R)_y \big\rangle_{R,y} \\ &= \int_{-\infty}^{+\infty} d_R^n y\, (u_{\bar{L}})_y(x^i) \overset{y}{\circledast} (c_R)_y = (c_R)_x, \end{aligned} \tag{84}$$

and

$$\begin{aligned}\psi(x^i) &= (-1)^n \big\langle (c'_L)_y, (\bar{u}_L)_y(x^i) \big\rangle'_{L,y} \\ &= \int_{-\infty}^{+\infty} d_L^n y\, (c'_L)_y \overset{y}{\circledast} (u_{\bar{R}})_y(x^i) = (c'_L)_x, \end{aligned} \tag{85}$$

$$\begin{aligned}\psi(x^i) &= (-1)^n \big\langle (c'_R)_y, (\bar{u}_L)_y(x^i) \big\rangle'_{R,y} \\ &= \int_{-\infty}^{+\infty} d_R^n y\, (c'_R)_y \overset{y}{\circledast} (u_{\bar{L}})_y(x^i) = (c'_R)_x. \end{aligned} \tag{86}$$

These relations can be checked in the following manner:

$$\begin{aligned}&\int_{-\infty}^{+\infty} d_L^n y\, (c'_L)_y \overset{y}{\circledast} (\bar{u}_{\bar{R}})_y(x^i) \\ &\quad= (\mathrm{vol}_{\bar{R}})^{-1} \int_{-\infty}^{+\infty} d_L^n y \int_{-\infty}^{+\infty} d_L^n \tilde{x}\, \psi(\tilde{x}^j) \overset{\tilde{x}}{\circledast} (u_{\bar{R}})_y(\tilde{x}^k) \overset{y}{\circledast} (\bar{u}_{\bar{R}})_y(x^i) \\ &\quad= (\mathrm{vol}_{\bar{R}})^{-2} \int_{-\infty}^{+\infty} d_L^n y \int_{-\infty}^{+\infty} d_L^n \tilde{x}\, \psi(\tilde{x}^j) \overset{\tilde{x}}{\circledast} \delta_{\bar{R}}^n(\tilde{x}^k \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} y^l)) \\ &\qquad\qquad \overset{\tilde{x}}{\circledast} \delta_{\bar{R}}^n(y^m \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} x^i)) \\ &\quad= (\mathrm{vol}_{\bar{R}})^{-1} \int_{-\infty}^{+\infty} d_L^n \tilde{x}\, \psi(\tilde{x}^j) \overset{\tilde{x}}{\circledast} \delta_{\bar{R}}^n(\tilde{x}^k \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} x^i)) = \psi(x^i). \end{aligned} \tag{87}$$

The expansions in (83)–(86) correspond to the completeness relations

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d_L^n y \, (u_{\bar{R}})_y(\tilde{x}^i) \overset{y}{\circledast} (\bar{u}_y)_{\bar{R}}(x^k) \\
&\qquad = (-1)^n \big\langle (\bar{u}_L)_y(\tilde{x}^i), (\bar{u}_y)_{\bar{R}}(x^k) \big\rangle_{L,y} \\
&\qquad = (-1)^n \big\langle (u_{\bar{R}})_y(\tilde{x}^i), (u_{\bar{L}})_y(x^k) \big\rangle'_{L,y} \\
&\qquad = (\mathrm{vol}_{\bar{R}})^{-1} \delta_{\bar{R}}^n(\tilde{x}^i \oplus_{\bar{R}} (\ominus_{\bar{R}} \, \kappa^{-1} x^k)),
\end{aligned} \tag{88}$$

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d_R^n y \, (u_{\bar{L}})_y(\,\tilde{x}^i) \overset{y}{\circledast} (\bar{u}_{\bar{L}})_y(x^k) \\
&\qquad = (-1)^n \big\langle (\bar{u}_R)_y(\tilde{x}^i), (\bar{u}_{\bar{L}})_y(x^k) \big\rangle_{R,x} \\
&\qquad = (-1)^n \big\langle (u_{\bar{L}})_y(\tilde{x}^i), (u_{\bar{R}})_y(x^k) \big\rangle'_{R,y} \\
&\qquad = (\mathrm{vol}_{\bar{L}})^{-1} \delta_{\bar{L}}^n(\tilde{x}^i \oplus_{\bar{L}} (\ominus_{\bar{L}} \, \kappa x^k)).
\end{aligned} \tag{89}$$

Last but not least, let us mention that the results of this subsection remain valid if we apply the substitutions in (47) and (48). For this case, however, the sesquilinear forms must be replaced according to the substitutions

$$\begin{aligned}
&\big\langle \,.\,,.\, \big\rangle_{L,x} \to \big\langle \,.\,,.\, \big\rangle_{1,x}, \quad \big\langle \,.\,,.\, \big\rangle_{R,x} \to \big\langle \,.\,,.\, \big\rangle_{2,x}, \\
&\big\langle \,.\,,.\, \big\rangle'_{L,x} \to \big\langle \,.\,,.\, \big\rangle'_{1,x}, \quad \big\langle \,.\,,.\, \big\rangle'_{R,x} \to \big\langle \,.\,,.\, \big\rangle'_{2,x}.
\end{aligned} \tag{90}$$

Finally, we have to take into account that the position eigenfunctions are represented by delta functions with a tilde and their conjugation properties require to drop the minus signs in front of the sesquilinear forms appearing in (79)-(86) and (88)-(89).

3.3 Orthonormality of position and momentum eigenfunctions

In the last subsection we saw that each wave function can be written as a linear combination of momentum or position eigenfunctions. For this reason momentum and position eigenfunctions satisfy completeness relations. In the case of momentum eigenfunctions the position variables x^i and the momentum variables p^i play the role of conjugate variables. The same holds for the variables x^i and y^i if we consider position eigenfunctions. Reversing the roles of conjugate variables (this can always be achieved since there is a complete symmetry between conjugate variables), we obtain from the

completeness relations in (73)-(74) and (88)-(89) orthonormality relations for momentum and position eigenfunctions, respectively.

For this to become more clear, we would like to demonstrate this procedure by an example. We start our considerations from the completeness relation (73):

$$\begin{aligned}
&(\mathrm{vol}_L)^{-1} \int_{-\infty}^{+\infty} d^n_{\bar{R}} p \, \exp(x^i|\mathrm{i}^{-1}p^k)_{\bar{R},L} \overset{p|y}{\odot}_R \exp(\ominus_{\bar{R}}\, y^j|\mathrm{i}^{-1}p^l)_{\bar{R},L} \\
&\quad = \int_{-\infty}^{+\infty} d^n_{\bar{R}} p \, (u_{\bar{R},L})_p(x^i) \overset{p|y}{\odot}_R (u_{\bar{R},L})_{\ominus_L p}(y^j) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-1} \delta^n_{\bar{R}}(x^i \oplus_{\bar{R}} (\ominus_{\bar{R}}\, y^j)). \qquad (91)
\end{aligned}$$

Interchanging the roles of position and momentum coordinates leads to

$$\begin{aligned}
&(\mathrm{vol}_L)^{-1} \int_{-\infty}^{+\infty} d^n_{\bar{R}} x \, \exp(\mathrm{i}^{-1}p^i|x^k)_{\bar{R},L} \overset{x|\tilde{p}}{\odot}_R \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L\, x^l)_{\bar{R},L} \\
&\quad = \int_{-\infty}^{+\infty} d^n_{\bar{R}} x \, (\bar{u}_{\bar{R},L})_p(x^k) \overset{x|\tilde{p}}{\odot}_R (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}}\tilde{p}}(x^l) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-1} \delta^n_{\bar{R}}(p^i \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \tilde{p}^j)). \qquad (92)
\end{aligned}$$

In this manner we get as orthonormality relations for momentum eigenfunctions

$$\begin{aligned}
&\big\langle (\bar{u}_{\bar{R},L})_p(x^k), (u_{\bar{R},L})_{\ominus_L\tilde{p}}(x^l) \big\rangle'_{\bar{R},x} = \big\langle (u_{\bar{R},L})_p(x^k), (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}}\tilde{p}}(x^l) \big\rangle_{\bar{R},x} \\
&\quad = \int_{-\infty}^{+\infty} d^n_{\bar{R}} x \, (\bar{u}_{\bar{R},L})_p(x^k) \overset{x|\tilde{p}}{\odot}_R (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}}\tilde{p}}(x^l) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{1/2} \mathcal{F}^*_{\bar{R}}((\bar{u}_{\bar{R},L})_p(x^k))(\tilde{p}^j) = (\mathrm{vol}_{\bar{R}})^{-1/2} \mathcal{F}_{\bar{R}}((\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}}p}(x^l))(p^i) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-1} \delta^n_{\bar{R}}(p^i \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \tilde{p}^j)), \qquad (93)
\end{aligned}$$

$$\begin{aligned}
&\big\langle (u_{R,\bar{L}})_{\ominus_{\bar{L}}\tilde{p}}(x^l), (\bar{u}_{R,\bar{L}})_p(x^k) \big\rangle'_{\bar{L},x} = \big\langle (\bar{u}_{R,\bar{L}})_{\ominus_R\tilde{p}}(x^l), (u_{R,\bar{L}})_p(x^k) \big\rangle_{\bar{L},x} \\
&\quad = \int_{-\infty}^{+\infty} d^n_{\bar{L}} p \, (u_{R,\bar{L}})_{\ominus_{\bar{L}}\tilde{p}}(x^l) \overset{\tilde{p}|x}{\odot}_L (u_{R,\bar{L}})_p(x^k) \\
&\quad = (\mathrm{vol}_{\bar{L}})^{1/2} \mathcal{F}^*_{\bar{L}}((u_{R,\bar{L}})_p(x^k))(\tilde{p}^j) = (\mathrm{vol}_{\bar{L}})^{-1/2} \mathcal{F}_{\bar{L}}((u_{R,\bar{L}})_{\ominus_{\bar{L}}p}(x^l))(p^i) \\
&\quad = (\mathrm{vol}_{\bar{L}})^{-1} \delta^n_{\bar{L}}((\ominus_{\bar{L}}\, \tilde{p}^j) \oplus_{\bar{L}} p^i). \qquad (94)
\end{aligned}$$

A similar method can be used to read off the corresponding orthonormality relations from the completeness relations for position eigenfunctions

. To begin with, we consider the completeness relation (88):

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d_L^n y\, (u_{\bar{R}})_y(\tilde{x}^i) \overset{y}{\circledast} (\bar{u}_{\bar{R}})_y(x^k) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-2} \int_{-\infty}^{+\infty} d_L^n y\, \delta_{\bar{R}}^n(\tilde{x}^i \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} y^j)) \overset{y}{\circledast} \delta_{\bar{R}}^n(y^l \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} x^k)) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-1} \delta_{\bar{R}}^n(\tilde{x}^i \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} x^k)).
\end{aligned} \tag{95}$$

From these identities it follows by renaming the variables that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d_L^n x\, (\bar{u}_{\bar{R}})_{\tilde{y}}(x^j) \overset{x}{\circledast} (u_{\bar{R}})_y(x^l) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-2} \int_{-\infty}^{+\infty} d_L^n x\, \delta_{\bar{R}}^n(\tilde{y}^i \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} x^j)) \overset{x}{\circledast} \delta_{\bar{R}}^n(x^l \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} y^k)) \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-1} \delta_{\bar{R}}^n(\tilde{y}^i \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} y^k)).
\end{aligned} \tag{96}$$

Proceeding this way we obtain as orthonormality relations

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d_L^n x\, (\bar{u}_{\bar{R}})_{\tilde{y}}(x^j) \overset{x}{\circledast} (u_{\bar{R}})_y(x^l) \\
&\quad = (-1)^n \big\langle (\bar{u}_{\bar{R}})_{\tilde{y}}(x^j), (\bar{u}_L)_y(x^l) \big\rangle'_{L,x} \\
&\quad = (-1)^n \big\langle (u_L)_{\tilde{y}}(x^j), (u_{\bar{R}})_y(x^l) \big\rangle_{L,x} \\
&\quad = (\mathrm{vol}_{\bar{R}})^{-1} \delta_{\bar{R}}^n(\tilde{y}^i \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \kappa^{-1} y^k)),
\end{aligned} \tag{97}$$

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d_R^n x\, (\bar{u}_{\bar{L}})_{\tilde{y}}(x^k) \overset{x}{\circledast} (u_{\bar{L}})_y(x^l) \\
&\quad = (-1)^n \big\langle (\bar{u}_{\bar{L}})_{\tilde{y}}(x^j), (\bar{u}_R)_y(x^l) \big\rangle'_{R,x} \\
&\quad = (-1)^n \big\langle (u_R)_{\tilde{y}}(x^j), (u_{\bar{L}})_y(x^j) \big\rangle_{R,x} \\
&\quad = (\mathrm{vol}_{\bar{L}})^{-1} \delta_{\bar{L}}^n(\tilde{y}^i \oplus_{\bar{L}} (\ominus_{\bar{L}}\, \kappa y^k)).
\end{aligned} \tag{98}$$

Finally, it should be clear that we are again allowed to apply the substitutions in (47), (48), and (90).

3.4 Operators in position and momentum basis

In Sec. 3.2 we concerned ourselves with expansions of wave functions in terms of position and momentum eigenfunctions. The corresponding coeffi-

cients gave us representations of wave functions in a position or momentum basis. In this subsection we would like to extend these reasonings to the expressions

$$\begin{aligned} P^k \triangleright \psi(x^j) &= \mathrm{i}\partial^k \triangleright \psi(x^j), & \hat{P}^k \triangleright \psi(x^j) &= \mathrm{i}\hat{\partial}^k \triangleright \psi(x^j), \\ P^k \,\bar{\triangleright}\, \psi(x^j) &= \mathrm{i}\partial^k \,\bar{\triangleright}\, \psi(x^j), & \hat{P}^k \,\bar{\triangleright}\, \psi(x^j) &= \mathrm{i}\hat{\partial}^k \,\bar{\triangleright}\, \psi(x^j), \end{aligned} \tag{99}$$

$$\begin{aligned} \psi(x^j) \triangleleft P^k &= \psi(x^j) \triangleleft (\mathrm{i}\partial^k), & \psi(x^j) \triangleleft \hat{P}^k &= \psi(x^j) \triangleleft (\mathrm{i}\hat{\partial}^k), \\ \psi(x^j) \,\bar{\triangleleft}\, P^k &= \psi(x^j) \,\bar{\triangleleft}\, (\mathrm{i}\partial^k), & \psi(x^j) \,\bar{\triangleleft}\, \hat{P}^k &= \psi(x^j) \,\bar{\triangleleft}\, (\mathrm{i}\hat{\partial}^k), \end{aligned} \tag{100}$$

and

$$X^k \triangleright \psi(x^j) = x^k \circledast \psi(x^j), \qquad \psi(x^j) \circledast x^k = \psi(x^j) \triangleleft X^k. \tag{101}$$

To begin with, we consider the Fourier transforms

$$\begin{aligned} &\mathcal{F}_L(P^m \overset{x}{\triangleright} \exp(\ominus_{\bar{R}}\, x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L})(p^i) \\ &\quad = \int_{-\infty}^{+\infty} d_L^n x\, (P^m \overset{x}{\triangleright} \exp(\ominus_{\bar{R}}\, x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L}) \overset{\tilde{p}|x}{\odot}_R \exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L} \\ &\quad = -\int_{-\infty}^{+\infty} d_L^n x\, (\tilde{p}^m \overset{\tilde{p}|x}{\odot}_R \exp(\ominus_{\bar{R}}\, x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L}) \overset{\tilde{p}|x}{\odot}_R \exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L} \\ &\quad = -\kappa\tilde{p}^m \overset{\tilde{p}}{\circledast} \delta_L^n((\ominus_L \tilde{p}^j) \oplus_L p^i), \end{aligned} \tag{102}$$

$$\begin{aligned} &\mathcal{F}_R(\exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_{\bar{L}}\, x^k)_{R,\bar{L}} \overset{x}{\triangleleft} \hat{P}^m)(p^i) \\ &\quad = \int_{-\infty}^{+\infty} d_R^n x\, \exp(\mathrm{i}^{-1}p^i|x^l)_{R,\bar{L}} \overset{x|\tilde{p}}{\odot}_L (\exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_{\bar{L}}\, x^k)_{R,\bar{L}} \overset{x}{\triangleleft} \hat{P}^m) \\ &\quad = -\int_{-\infty}^{+\infty} d_R^n x\, \exp(\mathrm{i}^{-1}p^i|x^l)_{R,\bar{L}} \overset{x|\tilde{p}}{\odot}_L (\exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_{\bar{L}}\, x^k)_{R,\bar{L}} \overset{x|\tilde{p}}{\odot}_L \tilde{p}^m) \\ &\quad = \delta_R^n(p^i \oplus_R (\ominus_R \tilde{p}^j)) \overset{\tilde{p}}{\circledast} (-\kappa^{-1}\tilde{p}^m), \end{aligned} \tag{103}$$

and

$$\begin{aligned} &\mathcal{F}_L^*(P^m \overset{x}{\triangleright} \exp(x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L})(p^i) \\ &\quad = (\mathrm{vol}_L)^{-1} \int_{-\infty}^{+\infty} d_L^n x\, \exp(\ominus_{\bar{R}}\, x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L} \overset{p|x}{\odot}_{\bar{L}} (P^m \overset{x}{\triangleright} \exp(x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L}) \end{aligned}$$

$$= (\mathrm{vol}_L)^{-1}\int_{-\infty}^{+\infty} d_L^n x\ \exp(\ominus_{\bar{R}}\, x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L} \overset{p|x}{\odot}_{\bar{L}} (\exp(x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L} \overset{\tilde{p}}{\circledast} \tilde{p}^m)$$

$$= (\mathrm{vol}_L)^{-1}\delta_L^n((\ominus_L\, p^i) \oplus_L \tilde{p}^j) \overset{\tilde{p}}{\circledast} \tilde{p}^m, \tag{104}$$

$$\mathcal{F}_R^*(\exp(\mathrm{i}^{-1}\tilde{p}^k|x^k)_{R,\bar{L}} \overset{x}{\triangleleft} \hat{P}^m)(p^i)$$

$$= (\mathrm{vol}_R)^{-1}\int_{-\infty}^{+\infty} d_R^n x\, (\exp(\mathrm{i}^{-1}\tilde{p}^j|x^k)_{R,\bar{L}} \overset{x}{\triangleleft} \hat{P}^m) \overset{x|p}{\odot}_{\bar{R}} \exp(\mathrm{i}^{-1}p^i|\ominus_{\bar{L}}\, x^l)_{R,\bar{L}}$$

$$= (\mathrm{vol}_R)^{-1}\int_{-\infty}^{+\infty} d_R^n x\, (\tilde{p}^m \overset{\tilde{p}}{\circledast} \exp(\mathrm{i}^{-1}\tilde{p}^j|x^k)_{R,\bar{L}}) \overset{x|p}{\odot}_{\bar{R}} \exp(\mathrm{i}^{-1}p^i|\ominus_{\bar{L}}\, x^l)_{R,\bar{L}}$$

$$= (\mathrm{vol}_R)^{-1}\tilde{p}^m \overset{\tilde{p}}{\circledast} \delta_R^n(\tilde{p}^j \oplus_R (\ominus_R\, p^i)). \tag{105}$$

Notice that the second equality in each of the above calculations makes use of the fact that q-exponentials are eigenfunctions of momentum operators (see the discussion in Ref. [37]) and the last step is a consequence of the addition law for q-exponentials and the defining expressions of q-delta functions. In terms of momentum eigenfunctions the above identities become

$$\begin{aligned}(P_L)^m_{\tilde{p}p} &= \big\langle P^m \overset{x}{\triangleright} (u_{\bar{R},L})_{\ominus_L\tilde{p}}(x^l), (\bar{u}_{\bar{R},L})_p(x^k)\big\rangle'_{L,x} \\ &= \big\langle (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}}\tilde{p}}(x^l) \overset{x}{\triangleleft} P_m, (u_{\bar{R},L})_p(x^k)\big\rangle_{L,x} \\ &= -(\mathrm{vol}_L)^{-1}\kappa\tilde{p}^m \overset{\tilde{p}}{\circledast} \delta_L^n((\ominus_L\, \tilde{p}^j) \oplus_L p^i),\end{aligned} \tag{106}$$

$$\begin{aligned}(P_R)^m_{p\tilde{p}} &= \big\langle (\bar{u}_{R,\bar{L}})_p(x^k), \hat{P}_m \overset{x}{\triangleright} (u_{R,\bar{L}})_{\ominus_{\bar{L}}\tilde{p}}(x^l)\big\rangle'_{R,x} \\ &= \big\langle (u_{R,\bar{L}})_p(x^k), (\bar{u}_{R,\bar{L}})_{\ominus_R\tilde{p}}(x^l) \overset{x}{\triangleleft} \hat{P}^m\big\rangle_{R,x} \\ &= -(\mathrm{vol}_R)^{-1}\kappa^{-1}\delta_{\bar{R}}^n(p^i \oplus_R (\ominus_R\, \tilde{p}^j)) \overset{\tilde{p}}{\circledast} \tilde{p}^m,\end{aligned} \tag{107}$$

and, likewise,

$$\begin{aligned}(P_L^*)^m_{p\tilde{p}} &= \big\langle (u_{\bar{R},L})_{\ominus_L p}(x^l), (\bar{u}_{\bar{R},L})_{\tilde{p}}(x^k) \overset{x}{\triangleleft} P_m\big\rangle'_{L,x} \\ &= \big\langle (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}} p}(x^l), P^m \overset{x}{\triangleright} (u_{\bar{R},L})_{\tilde{p}}(x^k)\big\rangle_{L,x} \\ &= (\mathrm{vol}_L)^{-1}\delta_L^n((\ominus_L\, p^i) \oplus_L \tilde{p}^j) \overset{\tilde{p}}{\circledast} \tilde{p}^m,\end{aligned} \tag{108}$$

$$(P_R^*)^m_{\tilde{p}p} = \big\langle (\bar{u}_{R,\bar{L}})_{\tilde{p}}(x^k) \overset{x}{\triangleleft} P^m, (u_{R,\bar{L}})_{\ominus_R p}(x^l)\big\rangle'_{R,x}$$

$$= \big\langle \hat{P}_m \overset{x}{\triangleright} (u_{R,\bar{L}})_{\tilde{p}}(x^k), (\bar{u}_{R,\bar{L}})_{\ominus_{\bar{L}} p}(x^l) \big\rangle_{R,x}$$
$$= (\mathrm{vol}_{\bar{R}})^{-1} \tilde{p}^m \overset{\tilde{p}}{\circledast} \delta^n_R(\tilde{p}^j \oplus_R (\ominus_R p^i)). \tag{109}$$

This way, we found the matrix elements of momentum operators in a basis of momentum eigenfunctions.

With these matrix elements at hand the action of a momentum operator on a wave function can be written as

$$\begin{aligned}((\psi(x^i) \overset{x}{\triangleleft} P^m)_L)_p &= \mathcal{F}_L(\psi(x^i) \overset{x}{\triangleleft} P^m)(p^j) \\ &= -\int_{-\infty}^{+\infty} d^n_{\bar{R}}\tilde{p}\, \kappa\, (c_L)_{\kappa^2 \tilde{p}} \overset{\tilde{p}}{\circledast} (P_L)^m_{\tilde{p}(\kappa^{-1}p)} = (c_L)_p \overset{p}{\circledast} p^m,\end{aligned} \tag{110}$$

$$\begin{aligned}((\hat{P}^m \overset{x}{\triangleright} \psi(x^i))_R)_p &= \mathcal{F}_R(\hat{P}^m \overset{x}{\triangleright} \psi(x^i))(p^j) \\ &= -\int_{-\infty}^{+\infty} d^n_{\bar{L}}\tilde{p}\, \kappa^{-1} (P_R)^m_{(\kappa p)\tilde{p}} \overset{\tilde{p}}{\circledast} (c_R)_{\kappa^{-2}\tilde{p}} = p^m \overset{p}{\circledast} (c_R)_p,\end{aligned} \tag{111}$$

and

$$\begin{aligned}((P^m \overset{x}{\triangleright} \psi(x^i))^*_L)_p &= \mathcal{F}^*_L(P^m \overset{x}{\triangleright} \psi(x^i))(p^j) \\ &= \int_{-\infty}^{+\infty} d^n_{\bar{R}}\tilde{p}\, (P^*_L)^m_{p\tilde{p}} \overset{\tilde{p}}{\circledast} (c^*_L)_{\kappa^{-1}\tilde{p}} = \kappa p^m \overset{p}{\circledast} (c^*_L)_p,\end{aligned} \tag{112}$$

$$\begin{aligned}((\psi(x^i) \overset{x}{\triangleleft} \hat{P}^m)^*_R)_p &= \mathcal{F}^*_R(\psi(x^i) \overset{x}{\triangleleft} \hat{P}^m)(p^j) \\ &= \int_{-\infty}^{+\infty} d^n_{\bar{L}}\tilde{p}\, (c^*_R)_{\kappa\tilde{p}} \overset{\tilde{p}}{\circledast} (P^*_R)^m_{\tilde{p}p} = \kappa^{-1} (c^*_R)_p \overset{p}{\circledast} p^m.\end{aligned} \tag{113}$$

Notice that the above relations are in accordance with the identities in (21), (22), (25), and (26). This way, we see that on momentum space momentum operators reduce to multiplication operators.

For the sake of completeness, it should be mentioned that the matrix elements for a product of momentum operators are obtained by a kind of matrix multiplication, i.e.

$$\begin{aligned}((P^m \cdot P^k)_L)_{\tilde{p}p} &= \int_{-\infty}^{+\infty} d^n_{\bar{R}}p'\, \kappa^2 (P_L)^m_{\tilde{p}(\kappa p')} \overset{p'}{\circledast} (P_L)^k_{p'(\kappa^{-1}p)}, \\ ((P^m \cdot P^k)_R)_{\tilde{p}p} &= \int_{-\infty}^{+\infty} d^n_{\bar{L}}p'\, \kappa^{-2} (P_R)^m_{(\kappa\tilde{p})p'} \overset{p'}{\circledast} (P_R)^k_{(\kappa^{-1}p')p},\end{aligned} \tag{114}$$

and

$$
\begin{aligned}
((P^m\cdot P^k)^*_L)_{\tilde p p} &= \int_{-\infty}^{+\infty} d^n_{\bar R}p'\,(P^*_L)^m_{\tilde p p'} \overset{p'}{\circledast} (P^*_L)^k_{(\kappa^{-1}p')p},\\
((P^m\cdot P^k)^*_R)_{\tilde p p} &= \int_{-\infty}^{+\infty} d^n_{\bar L}p'\,(P^*_R)^m_{(\kappa\tilde p)p'} \overset{p'}{\circledast} (P^*_R)^k_{p'p}.
\end{aligned} \tag{115}
$$

These identities are obtained most easily by applying the relations (110)-(113) in succession:

$$
\begin{aligned}
&(\psi(x^i)\overset{x}{\triangleleft}(P^m\cdot P^k)_L)_p = \mathcal{F}_L((\psi(x^i)\overset{x}{\triangleleft}P^m)\overset{x}{\triangleleft}P^k)(p^j)\\
&\quad= \int_{-\infty}^{+\infty} d^n_{\bar R}p'\,\kappa\mathcal{F}_L(\psi(x^i)\overset{x}{\triangleleft}P^m)(\kappa^2 p'^k)\overset{p'}{\circledast}(P_L)^k_{p'(\kappa^{-1}p)}\\
&\quad= \int_{-\infty}^{+\infty} d^n_{\bar R}\tilde p\,(c_L)_{\kappa^2\tilde p}\overset{\tilde p}{\circledast}\int_{-\infty}^{+\infty} d^n_{\bar R}p'\,\kappa^2(P_L)^m_{\tilde p(\kappa p')}\overset{p'}{\circledast}(P_L)^k_{p'(\kappa^{-1}p)}\\
&\quad= \int_{-\infty}^{+\infty} d^n_{\bar R}\tilde p\,(c_L)_{\kappa^2\tilde p}\overset{\tilde p}{\circledast}((P^m\cdot P^k)_L)_{\tilde p p}.
\end{aligned} \tag{116}
$$

Next, we would like to derive the matrix elements of position operators in a basis of momentum eigenfunctions. To reach this goal, we need the Fourier transforms

$$
\begin{aligned}
&\mathcal{F}^*_L(x^m\overset{x}{\circledast}\exp(x^k|\mathrm{i}^{-1}\tilde p^j)_{\bar R,L})(p^i)\\
&= \frac{1}{\mathrm{vol}_L}\int_{-\infty}^{+\infty} d^n_L x\,\exp(\ominus_{\bar R}\,x^l|\mathrm{i}^{-1}p^i)_{\bar R,L}\overset{p|x}{\odot}_{\bar L}(x^m\overset{x}{\circledast}\exp(x^k|\mathrm{i}^{-1}\tilde p^j)_{\bar R,L})\\
&= \frac{1}{\mathrm{vol}_L}\int_{-\infty}^{+\infty} d^n_L x\,\exp(\ominus_{\bar R}\,x^l|\mathrm{i}^{-1}p^i)_{\bar R,L}\overset{p|x}{\odot}_{\bar L}(\exp(x^k|\mathrm{i}^{-1}\tilde p^j)_{\bar R,L}\overset{\tilde p}{\triangleleft}(\mathrm{i}\partial^m))\\
&= (\mathrm{vol}_L)^{-1}\,\delta^n_L((\ominus_L\,p^i)\oplus_L\tilde p^j)\overset{\tilde p}{\triangleleft}(\mathrm{i}\partial^m),
\end{aligned} \tag{117}
$$

$$
\begin{aligned}
&\mathcal{F}^*_R(\exp(\mathrm{i}^{-1}\tilde p^j|x^k)_{R,\bar L}\overset{x}{\circledast}x^m)(p^i)\\
&= \frac{1}{\mathrm{vol}_R}\int_{-\infty}^{+\infty} d^n_R x\,(\exp(\mathrm{i}^{-1}\tilde p^j|x^k)_{R,\bar L}\overset{x}{\circledast}x^m)\overset{x|p}{\odot}_{\bar R}\exp(\mathrm{i}^{-1}p^i|\ominus_{\bar L}\,x^l)_{R,\bar L}\\
&= \frac{1}{\mathrm{vol}_R}\int_{-\infty}^{+\infty} d^n_R x\,(\mathrm{i}\hat\partial^m\overset{\tilde p}{\triangleright}\exp(\mathrm{i}^{-1}\tilde p^j|x^k)_{R,\bar L})\overset{x|p}{\odot}_{\bar R}\exp(\mathrm{i}^{-1}p^i|\ominus_{\bar L}\,x^l)_{R,\bar L}\\
&= \frac{1}{\mathrm{vol}_R}\,\mathrm{i}\hat\partial^m\overset{\tilde p}{\triangleright}\delta^n_R(\tilde p^j\oplus_R(\ominus_R\,p^i)),
\end{aligned} \tag{118}
$$

and

$$
\begin{aligned}
&\mathcal{F}_L(\exp(\ominus_{\bar{R}}\, x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L} \overset{\tilde{p}|x}{\odot}_R x^m)(p^i) \\
&\quad = \int_{-\infty}^{+\infty} d_L^n x\, (\exp(\ominus_{\bar{R}}\, x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L} \overset{\tilde{p}|x}{\odot}_R x^m) \overset{\tilde{p}|x}{\odot}_R \exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L} \\
&\quad = -\int_{-\infty}^{+\infty} d_L^n x\, (\exp(\ominus_{\bar{R}}\, x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L} \overset{\tilde{p}}{\triangleleft} (\mathrm{i}\partial^m)) \overset{\tilde{p}|x}{\odot}_R \exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L} \\
&\quad = \kappa^{-1}\mathrm{i}\partial^m \overset{\tilde{p}}{\triangleright} \delta_L^n((\ominus_L \tilde{p}^j) \oplus_L p^i),
\end{aligned}
\tag{119}
$$

$$
\begin{aligned}
&\mathcal{F}_R(x^m \overset{x|\tilde{p}}{\odot}_L \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_{\bar{L}}\, x^k)_{R,\bar{L}})(p^i) \\
&\quad = \int_{-\infty}^{+\infty} d_R^n x\, \exp(\mathrm{i}^{-1}p^i|x^l)_{R,\bar{L}} \overset{x|\tilde{p}}{\odot}_L (x^m \overset{x|\tilde{p}}{\odot}_L \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_{\bar{L}}\, x^k)_{R,\bar{L}}) \\
&\quad = -\int_{-\infty}^{+\infty} d_R^n x\, \exp(\mathrm{i}^{-1}p^i|x^l)_{R,\bar{L}} \overset{x|\tilde{p}}{\odot}_L (\mathrm{i}\hat{\partial}^m \overset{\tilde{p}}{\triangleright} \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_{\bar{L}}\, x^k)_{R,\bar{L}}) \\
&\quad = \kappa\delta_R^n(p^i \oplus_R (\ominus_R \tilde{p}^j)) \overset{\tilde{p}}{\triangleleft} (\mathrm{i}\hat{\partial}^m).
\end{aligned}
\tag{120}
$$

From the results in (117)-(120) we read off as matrix elements of position operators

$$
\begin{aligned}
(X_L^*)^m_{p\tilde{p}} &= \big\langle (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}}p}(x^l), x^m \overset{x}{\circledast} (u_{\bar{R},L})_{\tilde{p}}(x^k)\big\rangle_{L,x} \\
&= \big\langle (u_{\bar{R},L})_{\ominus_L p}(x^l), (\bar{u}_{\bar{R},L})_{\tilde{p}}(x^k) \overset{x}{\circledast} x_m\big\rangle'_{L,x} \\
&= (\mathrm{vol}_L)^{-1}\delta_L^n((\ominus_L p^i) \oplus_L \tilde{p}^j) \overset{\tilde{p}}{\triangleleft} (\mathrm{i}\partial^m),
\end{aligned}
\tag{121}
$$

$$
\begin{aligned}
(X_R^*)^m_{\tilde{p}p} &= \big\langle x_m \overset{x}{\circledast} (u_{R,\bar{L}})_{\tilde{p}}(x^k), (\bar{u}_{R,\bar{L}})_{\ominus_R p}(x^l)\big\rangle_{R,x} \\
&= \big\langle (\bar{u}_{R,\bar{L}})_{\tilde{p}}(x^k) \overset{x}{\circledast} x^m, (u_{R,\bar{L}})_{\ominus_{\bar{L}} p}(x^l),\big\rangle'_{R,x} \\
&= (\mathrm{vol}_{\bar{R}})^{-1}(\mathrm{i}\hat{\partial}^m) \overset{\tilde{p}}{\triangleright} \delta_R^n(\tilde{p}^j \oplus_R (\ominus_R p^i)),
\end{aligned}
\tag{122}
$$

and

$$
\begin{aligned}
(X_L)^m_{\tilde{p}p} &= \big\langle x_m \overset{x|\tilde{p}}{\odot}_{\bar{L}} (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}}\tilde{p}}(x^l), (u_{\bar{R},L})_p(x^k)\big\rangle_{L,x} \\
&= \big\langle (u_{\bar{R},L})_{\ominus_L\tilde{p}}(x^l) \overset{\tilde{p}|x}{\odot}_R x^m, (\bar{u}_{\bar{R},L})_p(x^k)\big\rangle'_{L,x}
\end{aligned}
$$

$$= \kappa^{-1}(\mathrm{vol}_L)^{-1}(\mathrm{i}\partial^m) \overset{\tilde{p}}{\triangleright} \delta_L^n((\ominus_L \tilde{p}^j) \oplus_L p^i), \tag{123}$$

$$\begin{aligned}(X_R)^m_{p\tilde{p}} &= \big\langle (u_{R,\bar{L}})_p(x^k), x^m \overset{x|\tilde{p}}{\odot}_L (\bar{u}_{R,\bar{L}})_{\ominus_R \tilde{p}}(x^l) \big\rangle_{R,x} \\ &= \big\langle (\bar{u}_{R,\bar{L}})_p(x^k), (u_{R,\bar{L}})_{\ominus_{\bar{L}} \tilde{p}}(x^l) \overset{\tilde{p}|x}{\odot}_{\bar{R}} x_m \big\rangle'_{R,x} \\ &= \kappa\, (\mathrm{vol}_R)^{-1}\, \delta_R^n(p^i \oplus_R (\ominus_R \tilde{p}^j)) \overset{\tilde{p}}{\triangleleft} (\mathrm{i}\partial^m).\end{aligned} \tag{124}$$

In complete analogy to (110)-(113) we obtain

$$\begin{aligned}((X^m \triangleright \psi(x^i))^*_L)_p &= \mathcal{F}^*_L(X^m \triangleright \psi(x^i))(p^j) = \mathcal{F}^*_L(x^m \overset{x}{\circledast} \psi(x^i))(p^j) \\ &= \int_{-\infty}^{+\infty} d^n_{\bar{R}}\tilde{p}\, (X^*_L)^m_{p\tilde{p}} \overset{\tilde{p}}{\circledast} (c^*_L)_{\kappa^{-1}\tilde{p}} = \kappa^{-1}\mathrm{i}\partial^m \overset{p}{\triangleright} (c^*_L)_p,\end{aligned} \tag{125}$$

$$\begin{aligned}((\psi(x^i) \overset{x}{\triangleleft} X^m)^*_R)_p &= \mathcal{F}^*_R(\psi(x^i) \overset{x}{\triangleleft} X^m)(p^j) = \mathcal{F}^*_R(\psi(x^i) \overset{x}{\circledast} x^m)(p^j) \\ &= \int_{-\infty}^{+\infty} d^n_{\bar{L}}\tilde{p}\, (c^*_R)_{\kappa\tilde{p}} \overset{\tilde{p}}{\circledast} (X^*_L)^m_{\tilde{p}p} = \kappa\, (c^*_R)_p \overset{p}{\triangleleft} (\mathrm{i}\hat{\partial}^m),\end{aligned} \tag{126}$$

and

$$\begin{aligned}((\psi(x^i) \overset{x}{\triangleleft} X^m)_L)_p &= \mathcal{F}_L(\psi(x^i) \overset{x}{\triangleleft} X^m)(p^j) = \mathcal{F}_L(\psi(x^i) \overset{x}{\circledast} x^m)(p^j) \\ &= \kappa^n \int_{-\infty}^{+\infty} d^n_{\bar{R}}\tilde{p}\, \kappa^{-1} (c_L)_{\kappa^2\tilde{p}} \overset{\tilde{p}}{\circledast} (X_L)^m_{\tilde{p}(\kappa^{-1}p)} = (c_L)_p \overset{p}{\triangleleft} (\mathrm{i}\partial^m),\end{aligned} \tag{127}$$

$$\begin{aligned}((X^m \triangleright \psi(x^i))_R)_p &= \mathcal{F}_R(X^m \triangleright \psi(x^i))(p^j) = \mathcal{F}_R(x^m \overset{x}{\circledast} \psi(x^i))(p^j) \\ &= \kappa^{-n} \int_{-\infty}^{+\infty} d^n_{\bar{L}}\tilde{p}\, \kappa\, (X_R)^m_{(\kappa p)\tilde{p}} \overset{\tilde{p}}{\circledast} (c_R)_{\kappa^{-2}\tilde{p}} = \mathrm{i}\hat{\partial}^m \overset{p}{\triangleright} (c_R)_p.\end{aligned} \tag{128}$$

Let us notice that these results are consistent with the identities in (21)-(24). The relations (125)-(128) tell us that on momentum space position operators act like derivatives.

Next, we would like to find representations of position and momentum operators in a basis of position eigenfunctions. This can be done most easily by applying the Fourier-Plancherel identities. With these identities at hand we get

$$\delta_L^n((\ominus_L p^i) \oplus_L \tilde{p}^j) \overset{\tilde{p}}{\circledast} \tilde{p}^m$$

$$
\begin{aligned}
&= \big\langle \exp(\mathrm{i}^{-1}p^i|\ominus_L x^l)_{\bar{R},L}, P^m \overset{x}{\triangleright} \exp(x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L}\big\rangle_{L,x} \\
&= (-1)^n\big\langle \mathcal{F}_{\bar{R}}(\exp(\mathrm{i}^{-1}p^i|\ominus_L x^l)_{\bar{R},L}(k^r)), \\
&\qquad\qquad \mathcal{F}^*_L(P^m \overset{x}{\triangleright} \exp(x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L})(\kappa^{-1}k^s)\big\rangle_{\bar{R},k} \\
&= (-1)^n\big\langle \mathcal{F}_{\bar{R}}(\exp(\mathrm{i}^{-1}p^i|\ominus_L x^l)_{\bar{R},L})(k^r), \\
&\qquad\qquad k^m \overset{k}{\circledast} \mathcal{F}^*_L(\exp(\kappa x^k|\mathrm{i}^{-1}\tilde{p}^j)_{\bar{R},L})(\kappa^{-1}k^s)\big\rangle_{\bar{R},k} \\
&= (-1)^n(\mathrm{vol}_L)^{-1}\big\langle \delta^n_{\bar{R}}(k^r \oplus_{\bar{R}} (\ominus_{\bar{R}}\, p^i)), \\
&\qquad\qquad k^m \overset{k}{\circledast} \delta^n_L((\ominus_L\, \kappa^{-1}k^s) \oplus_L \tilde{p}^j)\big\rangle_{\bar{R},k}.
\end{aligned} \tag{129}
$$

Let us make some comments on the above calculation. We start from (104) and use the Fourier-Plancherel identities [cf. the first identity in (45)]. Then we apply the fundamental properties of Fourier transformations [cf. the first identity in (26)]. For the last step we insert the expressions for Fourier transforms of q-exponentials [cf. the expressions in (29)-(32)]. With the same method we obtain from (105)

$$
\begin{aligned}
&\tilde{p}^m \overset{\tilde{p}}{\circledast} \delta^n_R(\tilde{p}^j \oplus_R (\ominus_R\, p^i)) \\
&\quad = \big\langle \exp(\mathrm{i}^{-1}\tilde{p}^j|x^k)_{R,\bar{L}} \overset{x}{\triangleleft} P^m, \exp(\ominus_R\, x^l|\mathrm{i}^{-1}p^i)_{R,\bar{L}}\big\rangle'_{R,x} \\
&\quad = (-1)^n(\mathrm{vol}_R)^{-1}\big\langle \delta^n_R(\tilde{p}^j \oplus_R (\ominus_R\, \kappa k^s)) \overset{k}{\circledast} k^m, \\
&\qquad\qquad\qquad \delta^n_{\bar{L}}((\ominus_{\bar{L}}\, p^i) \oplus_{\bar{L}} k^r)\big\rangle'_{\bar{L},k}.
\end{aligned} \tag{130}
$$

We can also start our considerations from (102) and (103). Proceeding in a similar fashion as above yields

$$
\begin{aligned}
&-\kappa\tilde{p}^m \overset{\tilde{p}}{\circledast} \delta^n_L((\ominus_L\, \tilde{p}^j) \oplus_L p^i) \\
&\quad = \big\langle \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L x^k)_{\bar{R},L} \overset{x}{\triangleleft} P_m, \exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L}\big\rangle_{L,x} \\
&\quad = (-1)^n\big\langle \mathcal{F}_{\bar{R}}(\exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L x^k)_{\bar{R},L} \overset{x}{\triangleleft} P_m)(k^r), \\
&\qquad\qquad \mathcal{F}^*_L(\exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L})(\kappa^{-1}k^s)\big\rangle_{\bar{R},k} \\
&\quad = (-1)^{n+1}\big\langle k_m \overset{k}{\circledast} \mathcal{F}_{\bar{R}}(\exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L x^k)_{\bar{R},L})(k^r), \\
&\qquad\qquad \mathcal{F}^*_L(\exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L})(\kappa^{-1}k^s)\big\rangle_{\bar{R},k} \\
&\quad = (-1)^{n+1}(\mathrm{vol}_L)^{-1}\big\langle \delta^n_{\bar{R}}(k^r \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \tilde{p}^j)),
\end{aligned}
$$

$$k^m \overset{k}{\circledast} \delta^n_L((\ominus_L \kappa^{-1} k^s) \oplus_L p^i)\big\rangle_{\bar{R},k}. \tag{131}$$

Likewise, we have

$$\begin{aligned}
&-\kappa^{-1}\delta^n_R(p^i \oplus_R (\ominus_R \tilde{p}^j)) \overset{\tilde{p}}{\circledast} \tilde{p}^m \\
&\quad= \big\langle \exp(\mathrm{i}^{-1}p^i|x^l)_{R,\bar{L}}, \hat{P}_m \overset{x}{\triangleright} \exp(\ominus_R x^k|\mathrm{i}^{-1}\tilde{p}^j)_{R,\bar{L}}\big\rangle'_{\bar{L},x} \\
&\quad= (-1)^{n+1}(\mathrm{vol}_R)^{-1}\big\langle \delta^n_R(p^i \oplus_R (\ominus_R \kappa k^s)) \overset{k}{\circledast} k^m, \\
&\qquad\qquad \delta^n_{\bar{L}}((\ominus_{\bar{L}} \tilde{p}^j) \oplus_{\bar{L}} k^r)\big\rangle'_{\bar{L},k}.
\end{aligned} \tag{132}$$

Further relations follow from the above results by applying the substitutions in (64).

To get expressions for matrix elements of position operators in a basis of position eigenfunctions we interchange the roles of position and momentum coordinates in the above relations and express the q-deformed delta functions in the sesquilinear forms by position eigenfunctions. This way, we should arrive at

$$\begin{aligned}
(X_{\bar{L}})^m_{y\tilde{y}} &= (-1)^n\big\langle (u_{\bar{L}})_y(x^r), x^m \overset{x}{\circledast} (u_R)_{\tilde{y}}(x^l)\big\rangle_{\bar{L},x} \\
&= (\mathrm{vol}_R)^{-1}\delta^n_R((\ominus_R \kappa y^i) \oplus_R \tilde{y}^j) \overset{\tilde{y}}{\circledast} \tilde{y}^m \\
&= (\mathrm{vol}_R)^{-1} y^m \overset{y}{\circledast} \delta^n_R(y^i \oplus_R (\ominus_R \kappa \tilde{y}^j)),
\end{aligned} \tag{133}$$

$$\begin{aligned}
(X_{\bar{R}})^m_{y\tilde{y}} &= (-1)^n\big\langle (u_{\bar{R}})_y(x^r), x^m \overset{x}{\circledast} (u_L)_{\tilde{y}}(x^l)\big\rangle_{\bar{R},x} \\
&= (\mathrm{vol}_L)^{-1}\,\delta^n_L((\ominus_L \kappa^{-1} y^i) \oplus_L \tilde{y}^j) \overset{\tilde{y}}{\circledast} \tilde{y}^m \\
&= (\mathrm{vol}_L)^{-1}\, y^m \overset{y}{\circledast} \delta^n_L(y^i \oplus_L (\ominus_L \kappa^{-1} \tilde{y}^j)),
\end{aligned} \tag{134}$$

and

$$\begin{aligned}
(X'_{\bar{L}})^m_{\tilde{y}y} &= (-1)^n\big\langle (\bar{u}_R)_{\tilde{y}}(x^l) \overset{x}{\circledast} x^m, (\bar{u}_y)_{\bar{L}}(x^r)\big\rangle'_{\bar{L},x} \\
&= (\mathrm{vol}_R)^{-1}\tilde{y}^m \overset{\tilde{y}}{\circledast} \delta^n_R(\tilde{y}^j \oplus_R (\ominus_R \kappa y^i)) \\
&= (\mathrm{vol}_R)^{-1}\delta^n_R((\ominus_R \kappa \tilde{y}^j) \oplus_R y^i) \overset{\tilde{y}}{\circledast} y^m,
\end{aligned} \tag{135}$$

$$(X'_{\bar{R}})^m_{y\tilde{y}} = (-1)^n\big\langle (\bar{u}_L)_{\tilde{y}}(x^l) \overset{x}{\circledast} x^m, (\bar{u}_y)_{\bar{R}}(x^r)\big\rangle'_{\bar{R},x}$$

$$
\begin{aligned}
&= (\mathrm{vol}_L)^{-1} \tilde{y}^m \overset{\tilde{y}}{\circledast} \delta_L^n(\tilde{y}^j \oplus_L (\ominus_L \kappa^{-1} y^i)) \\
&= (\mathrm{vol}_L)^{-1} \delta_L^n((\ominus_L \kappa^{-1} \tilde{y}^j) \oplus_L y^i) \overset{y}{\circledast} y^m. \qquad (136)
\end{aligned}
$$

As expected, in a basis of position eigenfunctions position operators act like multiplication operators, i.e.

$$
\begin{aligned}
((X^m \overset{x}{\triangleright} \psi(x^i))'_A)_y &= (-1)^n \big\langle (u_A)_y(x^j), X^m \overset{x}{\triangleright} \psi(x^i) \big\rangle_{A,x} \\
&= \int_{-\infty}^{+\infty} d_A^n \tilde{y}\, (X_A)^m_{y\tilde{y}} \overset{\tilde{y}}{\circledast} (c_A)_{\tilde{y}} = y^m \overset{y}{\circledast} (c_A)_y, \qquad (137)
\end{aligned}
$$

$$
\begin{aligned}
((\psi(x^i) \overset{x}{\triangleleft} X^m)'_A)_y &= (-1)^n \big\langle \psi(x^i) \overset{x}{\triangleleft} X^m, (\bar{u}_A)_y(x^j) \big\rangle'_{A,x} \\
&= \int_{-\infty}^{+\infty} d_A^n \tilde{y}\, (c'_A)_{\tilde{y}} \overset{\tilde{y}}{\circledast} (X'_A)^m_{\tilde{y}y} = (c'_A)_y \overset{y}{\circledast} y^m. \qquad (138)
\end{aligned}
$$

Lastly, we would like to close this subsection by constructing the matrix representation of momentum operators in a basis of position eigenfunctions. Once again, we use the Fourier-Plancherel identities to rewrite the matrix elements of position operators in a momentum basis. Reversing the roles of momentum and position variables will again enable us to read off the wanted matrix elements. Applying this procedure to the result of (117) yields

$$
\begin{aligned}
&\delta_L^n((\ominus_L p^i) \oplus_L \tilde{p}^j) \overset{\tilde{p}}{\triangleleft} (\mathrm{i}\partial^m) \\
&\quad = \big\langle \exp(\mathrm{i}^{-1} p^i | \ominus_L x^l)_{\bar{R},L}, x^m \overset{x}{\circledast} \exp(x^k | \mathrm{i}^{-1} \tilde{p}^j)_{\bar{R},L} \big\rangle_{L,x} \\
&\quad = (-1)^n \big\langle \mathcal{F}_{\bar{R}}(\exp(\mathrm{i}^{-1} p^i | \ominus_L x^l)_{\bar{R},L})(k^r), \\
&\qquad\qquad \mathcal{F}_L^*(x^m \overset{x}{\circledast} \exp(x^k | \mathrm{i}^{-1} \tilde{p}^j)_{\bar{R},L})(\kappa^{-1} k^s) \big\rangle_{\bar{R},k} \\
&\quad = (-1)^n \big\langle \mathcal{F}_{\bar{R}}(\exp(\mathrm{i}^{-1} p^i | \ominus_L x^l)_{\bar{R},L})(k^r), \\
&\qquad\qquad \mathrm{i}\partial^m \overset{k}{\triangleright} \mathcal{F}_L^*(\exp(x^k | \mathrm{i}^{-1} \tilde{p}^j)_{\bar{R},L})(\kappa^{-1} k^s) \big\rangle_{\bar{R},k} \\
&\quad = (-1)^n (\mathrm{vol}_L)^{-1} \big\langle \delta_{\bar{R}}^n(k^r \oplus_{\bar{R}} (\ominus_{\bar{R}} p^i)), \\
&\qquad\qquad \mathrm{i}\partial^m \overset{k}{\triangleright} \delta_L^n((\ominus_L \kappa^{-1} k^s) \oplus_L \tilde{p}^j) \big\rangle_{\bar{R},k}, \qquad (139)
\end{aligned}
$$

where we made use of the identities in (27), (41), and (42). Repeating the same steps for the result of (118) provides us with

$$
\mathrm{i}\hat{\partial}^m \overset{\tilde{p}}{\triangleright} \delta_R^n(\tilde{p}^j \oplus_R (\ominus_R p^i))
$$

$$= \big\langle \exp(\mathrm{i}^{-1}\tilde{p}^j|x^k)_{R,\bar{L}} \overset{x}{\circledast} x^m, \exp(\ominus_R\, x^l|\mathrm{i}^{-1}p^i)_{R,\bar{L}}\big\rangle'_{R,x}$$

$$= (-1)^n(\mathrm{vol}_R)^{-1}\big\langle \delta^n_R(\tilde{p}^j \oplus_R (\ominus_R\, \kappa k^s)) \overset{k}{\triangleleft} (\mathrm{i}\hat{\partial}^m), \\ \delta^n_{\bar{L}}((\ominus_{\bar{L}}\, p^i) \oplus_{\bar{L}} k^r)\big\rangle'_{\bar{L},k}. \tag{140}$$

Finally, the results of (119) and (120) respectively lead us to

$$\kappa^{-1}\mathrm{i}\partial^m \overset{\tilde{p}}{\triangleright} \delta^n_L((\ominus_L\, \tilde{p}^j) \oplus_L p^i) \\ = \big\langle x_m \overset{x|\tilde{p}}{\odot}_{\bar{L}} \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L\, x^k)_{\bar{R},L}, \exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L}\big\rangle_{L,x} \\ = (-1)^n\big\langle \mathcal{F}_{\bar{R}}(x_m \overset{x|\tilde{p}}{\odot}_{\bar{L}} \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L\, x^k)_{\bar{R},L})(k^r), \\ \mathcal{F}^*_L(\exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L})(\kappa^{-1}k^s)\big\rangle_{\bar{R},k} \\ = (-1)^n\big\langle i\partial_m \overset{k}{\triangleright} \mathcal{F}_{\bar{R}}(\exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L\, x^k)_{\bar{R},L})(k^r), \\ \mathcal{F}^*_L(\exp(x^l|\mathrm{i}^{-1}p^i)_{\bar{R},L})(\kappa^{-1}k^s)\big\rangle_{\bar{R},k} \\ = (-1)^{n+1}(\mathrm{vol}_L)^{-1}\big\langle \delta^n_{\bar{R}}(k^r \oplus_{\bar{R}} (\ominus_{\bar{R}}\, \tilde{p}^j)), \\ \mathrm{i}\partial_m \overset{k}{\triangleright} \delta^n_L((\ominus_L\, \kappa^{-1}k^s) \oplus_L p^i)\big\rangle_{\bar{R},k}, \tag{141}$$

and

$$\kappa\delta^n_R(p^i \oplus_R (\ominus_R\, \tilde{p}^j)) \overset{\tilde{p}}{\triangleleft} (\mathrm{i}\hat{\partial}^m) \\ = \big\langle \exp(\mathrm{i}^{-1}p^i|x^l)_{R,\bar{L}}, \exp(\ominus_R\, x^k|\mathrm{i}^{-1}\tilde{p}^j)_{R,\bar{L}} \overset{\tilde{p}|x}{\odot}_{\bar{R}} x_m\big\rangle'_{R,x} \\ = (-1)^n(\mathrm{vol}_R)^{-1}\big\langle \delta^n_R(p^i \oplus_R (\ominus_R\, \kappa k^s)) \overset{k}{\triangleleft} (\mathrm{i}\hat{\partial}^m), \\ \delta^n_{\bar{L}}((\ominus_{\bar{L}}\, \tilde{p}^j) \oplus_{\bar{L}} k^r)\big\rangle'_{\bar{L},k}. \tag{142}$$

Notice that the derivation in (141) requires to know the relation

$$\mathcal{F}_{\bar{R}}(x^m \overset{x|\tilde{p}}{\odot}_{\bar{L}} \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L\, x^l)_{\bar{R},L})(k^r) \\ = \mathrm{i}\partial^m \overset{k}{\triangleright} \mathcal{F}_{\bar{R}}(\exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L\, x^l)_{\bar{R},L})(k^r), \tag{143}$$

which can be proven in the following manner:

$$\mathcal{F}_{\bar{R}}(x^m \overset{x|\tilde{p}}{\odot}_{\bar{L}} \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L\, x^l)_{\bar{R},L})(k^r)$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} d^n_{\bar{R}}x \, (\exp(\mathrm{i}^{-1}k^r|x^j)_{\bar{R},L} \overset{x|\tilde{p}}{\odot}_{\bar{L}} (x^m \overset{x|\tilde{p}}{\odot}_{\bar{L}} \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L x^l)_{\bar{R},L}) \\
&= \int_{-\infty}^{+\infty} d^n_{\bar{R}}x \, (\exp(\mathrm{i}^{-1}k^r|x^j)_{\bar{R},L} \overset{x}{\circledast} x^m) \overset{x|\tilde{p}}{\odot}_{\bar{L}} \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L x^l)_{\bar{R},L} \\
&= \int_{-\infty}^{+\infty} d^n_{\bar{R}}x \, \mathrm{i}\partial^m \overset{k}{\triangleright} \exp(\mathrm{i}^{-1}k^r|x^j)_{\bar{R},L} \overset{x|\tilde{p}}{\odot}_{\bar{L}} \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L x^l)_{\bar{R},L} \\
&= \mathrm{i}\partial^m \overset{k}{\triangleright} \int_{-\infty}^{+\infty} d^n_{\bar{R}}x \, \exp(\mathrm{i}^{-1}k^r|x^j)_{\bar{R},L} \overset{x|\tilde{p}}{\odot}_{\bar{L}} \exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L x^l)_{\bar{R},L} \\
&= \mathrm{i}\partial^m \overset{k}{\triangleright} \mathcal{F}_{\bar{R}}(\exp(\mathrm{i}^{-1}\tilde{p}^j|\ominus_L x^l)_{\bar{R},L})(k^r). \qquad (144)
\end{aligned}
$$

Modifying the above results by the substitutions in (64) yields the corresponding relations for the other q-geometries.

Now, we are ready to write down matrix elements for momentum operators in a basis of position eigenfunctions:

$$
\begin{aligned}
(\hat{P}_{\bar{L}})^m_{y\tilde{y}} &= (-1)^n \big\langle (u_{\bar{L}})_y(x^r), P^m \overset{x}{\triangleright} (u_R)_{\tilde{y}}(x^l) \big\rangle_{\bar{L},x} \\
&= (\mathrm{vol}_R)^{-1} \mathrm{i}\hat{\partial}^m \overset{y}{\triangleright} \delta^n_R((\ominus_R \kappa y^i) \oplus_R \tilde{y}^j) \\
&= (\mathrm{vol}_R)^{-1} \, \delta^n_R((\ominus_R \kappa y^i) \oplus_R \tilde{y}^j) \overset{\tilde{y}}{\triangleleft} (\mathrm{i}\hat{\partial}^m), \qquad (145)
\end{aligned}
$$

$$
\begin{aligned}
(P_{\bar{R}})^m_{y\tilde{y}} &= (-1)^n \big\langle (u_{\bar{R}})_y(x^r), P^m \overset{x}{\triangleright} (u_L)_{\tilde{y}}(x^l) \big\rangle_{\bar{R},x} \\
&= (\mathrm{vol}_L)^{-1} \delta^n_L((\ominus_L \kappa^{-1} y^i) \oplus_L \tilde{y}^j) \overset{\tilde{y}}{\triangleleft} (\mathrm{i}\partial^m) \\
&= (\mathrm{vol}_L)^{-1} \mathrm{i}\partial^m \overset{y}{\triangleright} \delta^n_L((\ominus_L \kappa^{-1} y^i) \oplus_L \tilde{y}^j). \qquad (146)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(\hat{P}'_{\bar{L}})^m_{\tilde{y}y} &= (-1)^n \big\langle (\bar{u}_R)_{\tilde{y}}(x^l) \overset{x}{\triangleleft} \hat{P}^m, (\bar{u}_{\bar{L}})_y(x^r) \big\rangle'_{\bar{L},x} \\
&= (\mathrm{vol}_R)^{-1} \mathrm{i}\hat{\partial}^m \overset{\tilde{y}}{\triangleright} \delta^n_R(\tilde{y}^j \oplus_R (\ominus_R \kappa y^i)) \\
&= (\mathrm{vol}_R)^{-1} \delta^n_R(\tilde{y}^j \oplus_R (\ominus_R \kappa y^i)) \overset{y}{\triangleleft} (\mathrm{i}\hat{\partial}^m), \qquad (147)
\end{aligned}
$$

$$
\begin{aligned}
(P'_{\bar{R}})^m_{\tilde{y}y} &= (-1)^n \big\langle (\bar{u}_L)_{\tilde{y}}(x^l) \overset{x}{\triangleleft} P^m, (\bar{u}_{\bar{R}})_y(x^r) \big\rangle'_{\bar{R},x} \\
&= (\mathrm{vol}_L)^{-1} \delta^n_L(\tilde{y}^j \oplus_L (\ominus_L \kappa^{-1} y^i)) \overset{y}{\triangleleft} (\mathrm{i}\partial^m) \\
&= (\mathrm{vol}_L)^{-1} (\mathrm{i}\partial^m) \overset{\tilde{y}}{\triangleright} \delta^n_L(\tilde{y}^j \oplus_L (\ominus_L \kappa^{-1} y^i)). \qquad (148)
\end{aligned}
$$

Our results tell us that in position space momentum operators are represented by partial derivatives, i.e.

$$
\begin{aligned}
((\hat{P}^m \overset{x}{\bar{\triangleright}} \psi(x^i))_{\bar{L}})_y &= (-1)^n \big\langle (u_{\bar{L}})_y(x^j), \hat{P}^m \overset{x}{\bar{\triangleright}} \psi(x^i) \big\rangle_{\bar{L},x} \\
&= \int_{-\infty}^{+\infty} d^n_{\bar{L}}\tilde{y}\, (\hat{P}_{\bar{L}})^m_{y\tilde{y}} \overset{\tilde{y}}{\circledast} (c_{\bar{L}})_{\tilde{y}} = \mathrm{i}\hat{\partial}^m \overset{y}{\bar{\triangleright}} \psi(y^i), \qquad (149)
\end{aligned}
$$

$$
\begin{aligned}
((P^m \overset{x}{\bar{\triangleright}} \psi(x^i))_{\bar{R}})_y &= (-1)^n \big\langle (u_{\bar{R}})_y(x^j), P^m \overset{x}{\bar{\triangleright}} \psi(x^i) \big\rangle_{\bar{R},x} \\
&= \int_{-\infty}^{+\infty} d^n_{\bar{R}}\tilde{y}\, (P_{\bar{R}})^m_{y\tilde{y}} \overset{\tilde{y}}{\circledast} (c_{\bar{R}})_{\tilde{y}} = \mathrm{i}\partial^m \overset{y}{\bar{\triangleright}} \psi(y^i), \qquad (150)
\end{aligned}
$$

and

$$
\begin{aligned}
((\psi(x^i) \overset{x}{\bar{\triangleleft}} \hat{P}^m)'_{\bar{L}})_y &= (-1)^n \big\langle \psi(x^i) \overset{x}{\bar{\triangleleft}} \hat{P}^m, (\bar{u}_{\bar{L}})_y(x^j) \big\rangle'_{\bar{L},x} \\
&= \int_{-\infty}^{+\infty} d^n_{\bar{L}}\tilde{y}\, (c'_{\bar{L}})_{\tilde{y}} \overset{\tilde{y}}{\circledast} (\hat{P}'_{\bar{L}})^m_{\tilde{y}y} = \psi(y^i) \overset{y}{\bar{\triangleleft}} (\mathrm{i}\hat{\partial}^m). \qquad (151)
\end{aligned}
$$

$$
\begin{aligned}
((\psi(x^i) \overset{x}{\bar{\triangleleft}} P^m)'_{\bar{R}})_y &= (-1)^n \big\langle \psi(x^i) \overset{x}{\bar{\triangleleft}} P^m, (\bar{u}_{\bar{R}})_y(x^j) \big\rangle'_{\bar{R},x} \\
&= \int_{-\infty}^{+\infty} d^n_{\bar{R}}\tilde{y}\, (c'_{\bar{R}})_{\tilde{y}} \overset{\tilde{y}}{\circledast} (\hat{P}'_{\bar{R}})^m_{\tilde{y}y} = \psi(y^i) \overset{y}{\bar{\triangleleft}} (\mathrm{i}\partial^m). \qquad (152)
\end{aligned}
$$

Last but not least, let us note that the matrix elements for the other geometries are obtained from the results in this subsection most easily via the substitutions

$$
\begin{aligned}
&L \leftrightarrow \bar{L}, \quad R \leftrightarrow \bar{R}, \quad \triangleright \leftrightarrow \bar{\triangleright}, \quad \triangleleft \leftrightarrow \bar{\triangleleft}, \\
&\partial^i \leftrightarrow \hat{\partial}^i, \quad P^i \leftrightarrow \hat{P}^i, \quad \kappa \leftrightarrow \kappa^{-1}. \qquad (153)
\end{aligned}
$$

It should also be rather clear that we are allowed to apply the substitutions in (47), (48), and (90).

4 Spectral decomposition and projection operators

From the considerations of the previous section we know that wave functions can be expanded in terms of momentum or position eigenfunctions. Furthermore, we saw that momentum eigenfunctions as well as position eigenfunctions give not only a complete set of functions but also an or-

thonormal one. In the next section this observation will help us to bring our formalism in contact with quantum mechanics. To this aim it is useful to discuss a q-analog of the spectral decomposition of position and momentum operators.

First of all, let us recall that the spectral decomposition of a selfadjoint operator A is given by a sum or integral over products of eigenvalues with projection operators on the corresponding eigenspaces, i.e.

$$A = \int \lambda \, dP_\lambda + \sum_i \lambda_i P_i, \tag{154}$$

where P_i and dP_λ denote the projectors on the eigenspaces to the eigenvalues λ_i and λ, respectively. To find a q-analog of the spectral decomposition of position operators we need the identities in (18) and (19), which imply that

$$\begin{aligned} x^i \overset{x}{\circledast} f(x^j) &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x} \, \delta_B^n((\ominus_C \, \kappa_C x^k) \oplus_C \tilde{x}^l) \overset{\tilde{x}}{\circledast} \tilde{x}^i \overset{\tilde{x}}{\circledast} f(\tilde{x}^j), \\ f(x^j) \overset{x}{\circledast} x^i &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x} \, f(\tilde{x}^j) \overset{\tilde{x}}{\circledast} \tilde{x}^i \overset{\tilde{x}}{\circledast} \delta_B^n(\tilde{x}^l \oplus_C (\ominus_C \, \kappa_C x^k)). \end{aligned} \tag{155}$$

Since we require for the position operators X^i to act like multiplication operators, i.e.

$$X^i \overset{x}{\triangleright} f(x^j) = x^i \overset{x}{\circledast} f(x^j), \qquad f(x^j) \overset{x}{\triangleleft} X^i = f(x^j) \overset{x}{\circledast} x^i, \tag{156}$$

we conclude that

$$\begin{aligned} X^i \overset{x}{\triangleright} \ldots &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x} \, \delta_B^n((\ominus_C \, \kappa_C x^k) \oplus_C \tilde{x}^l) \overset{\tilde{x}}{\circledast} \tilde{x}^i \overset{\tilde{x}}{\circledast} \ldots, \\ \ldots \overset{x}{\triangleleft} X^i &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x} \, \ldots \overset{\tilde{x}}{\circledast} \tilde{x}^i \overset{\tilde{x}}{\circledast} \delta_B^n(x^l \oplus_C (\ominus_C \, \kappa_C x^k)). \end{aligned} \tag{157}$$

Let us make some comments on this result. It is important to realize that the variable $\tilde{x}^i$ represents the eigenvalues of the position operator X^i and the q-deformed delta functions behave like projectors on the corresponding eigenspaces. This way, we found a q-analog of the spectral decomposition of position operators.

It is not very difficult to extend the above formulae to functions of po-

sition operators. Obviously, we should have

$$
\begin{aligned}
F(X^i) \overset{x}{\triangleright} \ldots &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x}\, \delta_B^n((\ominus_C \kappa_C x^k) \oplus_C \tilde{x}^l) \overset{\tilde{x}}{\circledast} F(\tilde{x}^i) \overset{\tilde{x}}{\circledast} \ldots, \\
\ldots \overset{x}{\triangleleft} F(X^i) &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x}\; \ldots \overset{\tilde{x}}{\circledast} F(\tilde{x}^i) \overset{\tilde{x}}{\circledast} \delta_B^n(\tilde{x}^l \oplus_C (\ominus_C \kappa_C x^k)).
\end{aligned}
\tag{158}
$$

Especially, in the case $F(x^i) = 1$ we get

$$
\begin{aligned}
\mathrm{id} \overset{x}{\triangleright} \ldots &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x}\, \delta_B^n((\ominus_C \kappa_C x^k) \oplus_C \tilde{x}^l) \overset{\tilde{x}}{\circledast} \ldots, \\
\ldots \overset{x}{\triangleleft} \mathrm{id} &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x}\; \ldots \overset{\tilde{x}}{\circledast} \delta_B^n(\tilde{x}^l \oplus_C (\ominus_C \kappa_C x^k)).
\end{aligned}
\tag{159}
$$

Applying the expressions in (159) to a function on position space, we are able to expand this function in terms of position eigenfunctions. Thus, the formulae (159) can be viewed as completeness relations for position eigenfunctions. Once again, we see that the expansion coefficients are given by the function itself:

$$
\begin{aligned}
f(x^j) &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x}\, f(\tilde{x}^i) \overset{\tilde{x}}{\circledast} \delta_B^n(\tilde{x}^k \oplus_C (\ominus_C \kappa_C x^j)) \\
&= (-1)^n\, (\mathrm{vol}_{A,B})^{-1} \big\langle f(\tilde{x}^i), \overline{\delta_B^n(\tilde{x}^k \oplus_C (\ominus_C \kappa_C x^j))}\, \big\rangle'_{A,\tilde{x}},
\end{aligned}
\tag{160}
$$

$$
\begin{aligned}
f(\tilde{x}^j) &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n \tilde{x}\, \delta_B^n((\ominus_C \kappa_C x^j) \oplus_C \tilde{x}^k) \overset{\tilde{x}}{\circledast} f(\tilde{x}^i) \\
&= (-1)^n\, (\mathrm{vol}_{A,B})^{-1} \big\langle\, \overline{\delta_B^n((\ominus_C \kappa_C x^j) \oplus_C \tilde{x}^k)}, f(\tilde{x}^i) \big\rangle_{A,\tilde{x}}.
\end{aligned}
\tag{161}
$$

These observations are in complete accordance with the relations (79)-(82).

Let us recall that in position space eigenfunctions of position operators are given by delta functions. The scalar product between two q-deformed delta functions reads

$$
\begin{aligned}
&\frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n x\, \delta_B^n((\ominus_C \kappa_C y^i) \oplus_C x^k) \overset{x}{\circledast} \delta_B^n(x^l \oplus_C (\ominus_C \kappa_C \tilde{y}^j)) \\
&\quad = (\mathrm{vol}_{A,B})^{-1} \big\langle \delta_B^n((\ominus_C \kappa_C y^i) \oplus_C x^k), \overline{\delta_B^n(x^l \oplus_C (\ominus_C \kappa_C \tilde{y}^j))}\, \big\rangle'_{A,x} \\
&\quad = (\mathrm{vol}_{A,B})^{-1} \big\langle\, \overline{\delta_B^n((\ominus_C \kappa_C y^i) \oplus_C x^k)}, \delta_B^n(x^l \oplus_C (\ominus_C \kappa_C \tilde{y}^j)) \big\rangle_{A,x}
\end{aligned}
$$

$$= \delta_B^n((\ominus_C \kappa_C y^i) \oplus_C \tilde{y}^k) = \delta_B^n(y^l \oplus_C (\ominus_C \kappa_C \tilde{y}^j)). \tag{162}$$

These identities are consistent with the relations in (160) and (161) as can be seen by the calculations

$$\begin{aligned}
&\frac{1}{(\mathrm{vol}_{A,B})^2} \Big\langle \overline{\int_{-\infty}^{+\infty} d_A^n y\, \overline{f(y^m)} \overset{y}{\circledast} \delta_B^n((\ominus_C \kappa_C y^k) \oplus_C x^i)}, \\
&\qquad \int_{-\infty}^{+\infty} d_A^n \tilde{y}\, \delta_B^n(x^j \oplus_C (\ominus_C \kappa_C \tilde{y}^l)) \overset{\tilde{y}}{\circledast} g(\tilde{y}^r) \Big\rangle_{A,x} \\
&= \frac{1}{(\mathrm{vol}_{A,B})^2} \int_{-\infty}^{+\infty} d_A^n y \int_{-\infty}^{+\infty} d_A^n \tilde{y}\, \overline{f(y^m)} \overset{y}{\circledast} \big\langle \overline{\delta_B^n((\ominus_C \kappa_C y^k) \oplus_C x^i)}, \\
&\qquad \delta_B^n(x^j \oplus_C (\ominus_C \kappa_C \tilde{y}^l)) \big\rangle_{A,x} \overset{\tilde{y}}{\circledast} g(\tilde{y}^r) \\
&= \int_{-\infty}^{+\infty} d_A^n y\, \overline{f(y^m)} \overset{y}{\circledast} g(y^r) = \big\langle f(y^m), g(y^r) \big\rangle_{A,y},
\end{aligned} \tag{163}$$

and

$$\begin{aligned}
&\frac{1}{(\mathrm{vol}_{A,B})^2} \Big\langle \int_{-\infty}^{+\infty} d_A^n y\, f(y^m) \overset{y}{\circledast} \delta_B^n((\ominus_C \kappa_C y^k) \oplus_C x^i), \\
&\qquad \overline{\int_{-\infty}^{+\infty} d_A^n \tilde{y}\, \delta_B^n(x^j \oplus_C (\ominus_C \kappa_C \tilde{y}^l)) \overset{\tilde{y}}{\circledast} \overline{g(\tilde{y}^r)}} \Big\rangle'_{A,x} \\
&= \frac{1}{(\mathrm{vol}_{A,B})^2} \int_{-\infty}^{+\infty} d_A^n y \int_{-\infty}^{+\infty} d_A^n \tilde{y}\, f(y^m) \overset{y}{\circledast} \big\langle \delta_B^n((\ominus_C \kappa_C y^k) \oplus_C x^i), \\
&\qquad \overline{\delta_B^n(x^j \oplus_C (\ominus_C \kappa_C \tilde{y}^l))} \big\rangle'_{A,x} \overset{\tilde{y}}{\circledast} \overline{g(\tilde{y}^r)} \\
&= \int_{-\infty}^{+\infty} d_A^n y\, f(y^m) \overset{y}{\circledast} \overline{g(y^r)} = \big\langle f(y^m), g(y^r) \big\rangle_{A,y}.
\end{aligned} \tag{164}$$

In the above calculation we consider two functions f and g in position space and take the scalar product between their expansions in terms of position eigenfunctions. After some rearrangements, the orthonormality relation in (162) allows us to regain the scalar product between the two functions f and g.

Let us return to the spectral decomposition of position operators. A short look at (156) and (157) should make it obvious that position operators become diagonal in the representation space of position eigenfunctions. For the momentum operators this is not the case as becomes clear from the

identities

$$\begin{aligned}
\mathrm{i}\partial^i \overset{x}{\triangleright} f(x^j) &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n y \, \big\langle \, \overline{\delta_B^n((\ominus_C \, \kappa_C x^k) \oplus_C \tilde{x}^l)}, \\
&\qquad \mathrm{i}\partial^i \overset{\tilde{x}}{\triangleright} \delta_B^n(\tilde{x}^m \oplus_C (\ominus_C \, \kappa_C y^r)) \big\rangle_{A,\tilde{x}} \overset{y}{\circledast} f(y^j) \\
&= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n y \, \big\langle \delta_B^n((\ominus_C \, \kappa_C x^k) \oplus_C \tilde{x}^l), \\
&\qquad \overline{\mathrm{i}\partial^i \overset{\tilde{x}}{\triangleright} \delta_B^n(\tilde{x}^m \oplus_C (\ominus_C \, \kappa_C y^r))} \big\rangle'_{A,\tilde{x}} \overset{y}{\circledast} f(y^j),
\end{aligned} \tag{165}$$

$$\begin{aligned}
f(x^j) \overset{x}{\triangleleft} \mathrm{i}\partial^i &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n y \, f(y^j) \overset{y}{\circledast} \big\langle \, \overline{\delta_B^n((\ominus_C \, \kappa_C y^k) \oplus_C \tilde{x}^l) \overset{\tilde{x}}{\triangleleft} (\mathrm{i}\partial^i)}, \\
&\qquad \delta_B^n(\tilde{x}^m \oplus_C (\ominus_C \, \kappa_C x^r)) \big\rangle_{A,\tilde{x}} \\
&= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n y \, f(y^j) \overset{y}{\circledast} \big\langle \delta_B^n((\ominus_C \, \kappa_C y^k) \oplus_C \tilde{x}^l) \overset{\tilde{x}}{\triangleleft} (\mathrm{i}\partial^i), \\
&\qquad \overline{\delta_B^n(\tilde{x}^m \oplus_C (\ominus_C \, \kappa_C x^r))} \big\rangle'_{A,\tilde{x}}.
\end{aligned} \tag{166}$$

Thus, in the representation space of position eigenfunctions momentum operators act as integral operators:

$$\begin{aligned}
\mathrm{i}\partial^i \overset{x}{\triangleright} \ldots &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n y \, (\mathrm{i}\partial^i)^C_{A,B}(x^k, y^r) \overset{y}{\circledast} \ldots, \\
\ldots \overset{x}{\triangleleft} (\mathrm{i}\partial^i) &= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n y \, \ldots \overset{y}{\circledast} (\mathrm{i}\partial^i)^C_{A,B}(y^r, x^k),
\end{aligned} \tag{167}$$

where we introduced, for brevity, the matrix elements

$$\begin{aligned}
&(\mathrm{i}\partial^i)^C_{A,B}(x^k, y^r) \\
&\quad \equiv \big\langle \, \overline{\delta_B^n((\ominus_C \, \kappa_C x^k) \oplus_C \tilde{x}^l)}, \mathrm{i}\partial^i \overset{\tilde{x}}{\triangleright} \delta_B^n(\tilde{x}^m \oplus_C (\ominus_C \, \kappa_C y^r)) \big\rangle_{A,\tilde{x}} \\
&\quad = \big\langle \delta_B^n((\ominus_C \, \kappa_C x^k) \oplus_C \tilde{x}^l), \overline{\mathrm{i}\partial^i \overset{\tilde{x}}{\triangleright} \delta_B^n(\tilde{x}^m \oplus_C (\ominus_C \, \kappa_C y^r))} \big\rangle'_{A,\tilde{x}},
\end{aligned} \tag{168}$$

$$\begin{aligned}
&(\mathrm{i}\partial^i)^C_{A,B}(y^k, x^r) \\
&\quad \equiv \big\langle \, \overline{\delta_B^n((\ominus_C \, \kappa_C y^k) \oplus_C \tilde{x}^l) \overset{\tilde{x}}{\triangleleft} (\mathrm{i}\partial^i)}, \delta_B^n(\tilde{x}^m \oplus_C (\ominus_C \, \kappa_C x^r)) \big\rangle_{A,\tilde{x}} \\
&\quad = \big\langle \delta_B^n((\ominus_C \, \kappa_C y^k) \oplus_C \tilde{x}^l) \overset{\tilde{x}}{\triangleleft} (\mathrm{i}\partial^i), \overline{\delta_B^n(\tilde{x}^m \oplus_C (\ominus_C \, \kappa_C x^r))} \big\rangle'_{A,\tilde{x}}.
\end{aligned} \tag{169}$$

One should keep in mind that the considerations so far carry over to representations of momentum operators on momentum space. Since position and momentum variables play symmetrical roles in our approach, the results for momentum operators are obtained most easily from those for position operators by substituting position variables with momentum variables and vice versa. This way, we see that we have two representations, one in which position operators become diagonal and one which does the same for momentum operators. In complete analogy to the undeformed case, q-deformed Fourier transformations allow us to switch between the two representations and q-deformed exponentials can be seen as the corresponding transformation functions (or overlap matrix element between position and momentum eigenstates).

From functional analysis we know that the spectral decomposition of multiplication operators gives rise to a so-called spectral measure. A comparison of (155) with the classical spectral decomposition of multiplication operators shows us that a q-deformed delta function can play the role of a spectral measure:

$$\begin{aligned} \frac{dE^C_{A,B}(\tilde{x}^k, x^l)}{d^n_A\tilde{x}} &\equiv \frac{1}{\mathrm{vol}_{A,B}}\,\delta^n_B((\ominus_C\,\kappa_C\tilde{x}^k)\oplus_C x^l), \\ \frac{d\bar{E}^C_{A,B}(x^l, \tilde{x}^k)}{d^n_A\tilde{x}} &\equiv \frac{1}{\mathrm{vol}_{A,B}}\,\delta^n_B(x^l\oplus_C(\ominus_C\,\kappa_C\tilde{x}^k)). \end{aligned} \tag{170}$$

The quantities in (170) give operators that project onto eigenspaces to the eigenvalues $\tilde{x}^k$. Sometimes they are called *eigenfunctionals* or *eigendistributions*, since they give a complete and orthonormal set of solutions to the equations in (55) and (56).

Let us recall that a projector P has to fulfill the relation $P^2 = P$, i.e. it is idempotent. For the operations in (170) this property can be checked as follows:

$$\begin{aligned} &\int_{-\infty}^{+\infty} d^n_A y\,\frac{dE^C_{A,B}(\tilde{x}^i, y^k)}{d^n_A\tilde{x}} \overset{y}{\circledast} \frac{dE^C_{A,B}(y^l, x^j)}{d^n_A y} \\ &= \frac{1}{(\mathrm{vol}_{A,B})^2}\int_{-\infty}^{+\infty} d^n_A y\,\delta^n_B((\ominus_C\,\kappa_C\tilde{x}^i)\oplus_C y^k) \overset{y}{\circledast} \delta^n_B((\ominus_C\,\kappa_C y^l)\oplus_C x^j) \\ &= (\mathrm{vol}_{A,B})^{-1}\,\delta^n_B((\ominus_C\,\kappa_C\tilde{x}^i)\oplus_C x^j) = \frac{dE^C_{A,B}(\tilde{x}^i, x^j)}{d^n_A\tilde{x}}, \end{aligned} \tag{171}$$

and

$$\int_{-\infty}^{+\infty} d_A^n y \, \frac{d\bar{E}_{A,B}^C(\tilde{x}^i, y^k)}{d_A^n y} \overset{y}{\circledast} \frac{d\bar{E}_{A,B}^C(y^l, \tilde{x}^j)}{d_A^n \tilde{x}}$$
$$= \frac{1}{(\mathrm{vol}_{A,B})^2} \int_{-\infty}^{+\infty} d_A^n y \, \delta_B^n(x^i \oplus_C (\ominus_C \kappa_C y^k)) \overset{y}{\circledast} \delta_B^n(y^l \oplus_C (\ominus_C \kappa_C \tilde{x}^j))$$
$$= (\mathrm{vol}_{A,B})^{-1} \, \delta_B^n(x^i \oplus_C (\ominus_C \kappa_C \tilde{x}^j)) = \frac{d\bar{E}_{A,B}^C(x^i, \tilde{x}^j)}{d_A^n \tilde{x}}. \tag{172}$$

Applying q-deformed integrals to the operators in (170) we can construct further projectors given by

$$E_{A,B}^C(\tilde{x}^j, x^l) = \int_{-\infty}^{\tilde{x}^j} d_A^n y \, \frac{dE_{A,B}^C(y^k, x^l)}{d_A^n y}$$
$$= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{\tilde{x}^j} d_A^n y \, \delta_B^n((\ominus_C \kappa_C y^k) \oplus_C x^l), \tag{173}$$

$$\bar{E}_{A,B}^C(x^l, \tilde{x}^j) = \int_{-\infty}^{\tilde{x}^j} d_A^n y \, \frac{d\bar{E}_{A,B}^C(x^l, y^k)}{d_A^n y}$$
$$= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{\tilde{x}^j} d_A^n y \, \delta_B^n(x^l \oplus_C (\ominus_C \kappa_C y^k)). \tag{174}$$

For these projectors it holds

$$\int_{-\infty}^{+\infty} d_A^n x \, E_{A,B}^C(\tilde{x}^j, x^l) \overset{x}{\circledast} f(x^i)$$
$$= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n x \int_{-\infty}^{\tilde{x}^j} d_A^n y \, \delta_B^n((\ominus_C \kappa_C y^k) \oplus_C x^l) \overset{x}{\circledast} f(x^i)$$
$$= \int_{-\infty}^{\tilde{x}^j} d_A^n y \, f(y^i), \tag{175}$$

and

$$\int_{-\infty}^{+\infty} d_A^n x \, f(x^i) \overset{x}{\circledast} \bar{E}_{A,B}^C(x^l, \tilde{x}^j)$$
$$= \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{+\infty} d_A^n x \int_{-\infty}^{\tilde{x}^j} d_A^n y \, f(x^i) \overset{x}{\circledast} \delta_B^n(x^l \oplus_C (\ominus_C \kappa_C y^k))$$

$$= \int_{-\infty}^{\tilde{x}^j} d_A^n y\, f(y^i). \tag{176}$$

A short glance at the above calculations tells us that the operators in (173) and (174) can be viewed as a sum of projectors on eigenspaces whose eigenvalues are smaller than $\tilde{x}^k$. In this interpretation the q-deformed integrals take the role of the sum over different eigenvalues.

The operators in (173) and (174) can be related to q-analogs of the Heaviside function through

$$\begin{aligned}
\Theta^C_{A,B}(\ominus_C\, \kappa_C x^i) &= E^C_{A,B}(\tilde{x}^k = 0, x^i), \\
\bar{\Theta}^C_{A,B}(\ominus_C\, \kappa_C x^i) &= \bar{E}^C_{A,B}(x^i, \tilde{x}^k = 0),
\end{aligned} \tag{177}$$

if the q-deformed Heaviside functions are defined by

$$\begin{aligned}
\Theta^C_{A,B}(x^i) &\equiv \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{0} d_A^n y\, \delta_B^n(y^j \oplus_C x^i), \\
\bar{\Theta}^C_{A,B}(x^i) &\equiv \frac{1}{\mathrm{vol}_{A,B}} \int_{-\infty}^{0} d_A^n y\, \delta_B^n(x^i \oplus_C y^i).
\end{aligned} \tag{178}$$

Let us recall that in the so-called Dyson series for the time-evolution operator the appearance of Heaviside functions guarantees the principle of causality. In this respect the following modifications of the q-deformed Heaviside functions could prove useful:

$$\begin{aligned}
\Theta^R_{L,B}(\tilde{x}^i \oplus_R (\ominus_R\, x^j)) &= \frac{1}{\mathrm{vol}_{L,B}} \int_{-\infty}^{0} d_L^n y\, \delta_B^n(y^k \oplus_R \tilde{x}^i \oplus_R (\ominus_R\, x^j)) \\
&= \frac{1}{\mathrm{vol}_{L,B}} \int_{-\infty}^{\tilde{x}^i} d_L^n y\, \delta_B^n(y^k \oplus_R (\ominus_R\, x^j)),
\end{aligned} \tag{179}$$

$$\begin{aligned}
\bar{\Theta}^L_{R,B}((\ominus_L\, x^j) \oplus_L \tilde{x}^i) &= \frac{1}{\mathrm{vol}_{R,B}} \int_{-\infty}^{0} d_R^n y\, \delta_B^n((\ominus_L\, x^j) \oplus_L \tilde{x}^i \oplus_L y^k) \\
&= \frac{1}{\mathrm{vol}_{R,B}} \int_{-\infty}^{\tilde{x}^i} d_R^n y\, \delta_B^n((\ominus_L\, x^j) \oplus_L y^k).
\end{aligned} \tag{180}$$

Notice that the last equality in (179) as well as (180) is a consequence of

the relations (see Ref. [37])

$$\int_{-\infty}^{(y^j\oplus_L x^k)^i} d_R\tilde{x}^i\, f(\tilde{x}^l) = \int_{-\infty}^{x^i} d_R\tilde{x}^i\, f(y^j \oplus_L \tilde{x}^k),$$
$$\int_{-\infty}^{(x^k\oplus_R y^j)^i} d_L\tilde{x}^i\, f(\tilde{x}^l) = \int_{-\infty}^{x^i} d_L\tilde{x}^i\, f(\tilde{x}^k \oplus_R y^j), \tag{181}$$

since they imply

$$\int_{-\infty}^{y^i} d_{\bar{L}}\tilde{x}^i\, f(\tilde{x}^j) = \int_{-\infty}^{0} d_{\bar{L}}\tilde{x}^i\, f(\tilde{x}^k \oplus_R y^j),$$
$$\int_{-\infty}^{y^i} d_R\tilde{x}^i\, f(\tilde{x}^j) = \int_{-\infty}^{0} d_R\tilde{x}^i\, f(y^j \oplus_{\bar{L}} \tilde{x}^k). \tag{182}$$

As characteristic properties the q-deformed Heaviside functions in (179) and (180) show

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d_L^n x\, \Theta^R_{L,\bar{R}}(\tilde{x}^i \oplus_R (\ominus_R\, \kappa_R x^j)) \overset{x}{\circledast} f(x^k) \\
&\quad= \frac{1}{\mathrm{vol}_L}\int_{-\infty}^{+\infty} d_L^n x \int_{-\infty}^{\tilde{x}^i} d_L^n y\, \delta^n_{\bar{R}}(y^l \oplus_R (\ominus_R\, \kappa_R x^j)) \overset{x}{\circledast} f(x^k) \\
&\quad= \frac{1}{\mathrm{vol}_L}\int_{-\infty}^{\tilde{x}^i} d_L^n y \int_{-\infty}^{+\infty} d_L^n x\, \delta^n_{\bar{R}}((\ominus_R\, \kappa_R y^l) \oplus_R x^j) \overset{x}{\circledast} f(x^k) \\
&\quad= \int_{-\infty}^{\tilde{x}^i} d_L^n y\, f(y^l),
\end{aligned} \tag{183}$$

and

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d_R^n x\, f(x^k) \overset{x}{\circledast} \bar{\Theta}^L_{R,\bar{L}}((\ominus_L\, \kappa_L x^j) \oplus_L \tilde{x}^i) \\
&\quad= \frac{1}{\mathrm{vol}_L}\int_{-\infty}^{+\infty} d_R^n x \int_{-\infty}^{\tilde{x}^i} d_R^n y\, f(x^k) \overset{x}{\circledast} \delta^n_{\bar{L}}((\ominus_L\, \kappa_L x^j) \oplus_L y^l) \\
&\quad= \frac{1}{\mathrm{vol}_L}\int_{-\infty}^{\tilde{x}^i} d_R^n y \int_{-\infty}^{+\infty} d_R^n x\, f(x^k) \overset{x}{\circledast} \delta^n_{\bar{L}}(x^j \oplus_{\bar{L}} (\ominus_L\, \kappa_L y^l))
\end{aligned}$$

$$= \int_{-\infty}^{\tilde{x}^i} d^n_{\bar{R}} y\, f(y^l). \tag{184}$$

To gain further insight into the nature of eigenspaces and their eigenvalues, let us take a closer look at q-deformed integrals over the whole space. In our previous work (see Refs. [22,37]) we showed that for the quantum spaces we are interested in integrals over the whole space are given by products of Jackson integrals. Especially, we found

(i) (quantum plane)

$$\int_{-\infty}^{+\infty} d^2_L x\, f(x^1, x^2) = -\frac{q}{16}(D^1_{q^{1/2}})^{-1}\big|^{\infty}_{-\infty}(D^2_{q^{1/2}})^{-1}\big|^{\infty}_{-\infty} f, \tag{185}$$

(ii) (three-dimensional Euclidean space)

$$\int_{-\infty}^{+\infty} d^3_L x\, f(x^+, x^3, x^-)$$
$$= \frac{q^{-6}}{4}(D^+_{q^2})^{-1}\big|^{\infty}_{-\infty}(D^3_{q^2})^{-1}\big|^{\infty}_{-\infty}(D^-_{q^2})^{-1}\big|^{\infty}_{-\infty} f, \tag{186}$$

(iii) (four-dimensional Euclidean space)

$$\int_{-\infty}^{+\infty} d^4_L x\, f(x^4, x^3, x^2, x^1)$$
$$= \frac{1}{16}(D^1_q)^{-1}\big|^{\infty}_{-\infty}(D^2_q)^{-1}\big|^{\infty}_{-\infty}(D^3_q)^{-1}\big|^{\infty}_{-\infty}(D^4_q)^{-1}\big|^{\infty}_{-\infty} f, \tag{187}$$

(iv) (q-deformed Minkowski space)

$$\int_{-\infty}^{+\infty} d^4_L x\, f(r^2, x^-, x^{3/0}, x^+)$$
$$= -\frac{1}{16}(q\lambda_+)^3(D^{r^2}_{q^{-1}})^{-1}\big|^{\infty}_{-\infty}(D^+_{q^{-1}})^{-1}\big|^{\infty}_{-\infty}(D^{3/0}_{q^{-1}})^{-1}\big|^{\infty}_{-\infty}(D^-_{q^{-1}})^{-1}\big|^{\infty}_{-\infty} f, \tag{188}$$

where the *Jackson integrals* [52] are given by ($a > 0$, $1 \leq c < q^a$)

$$(D_{q^{\pm a}})^{-1}\big|_0^{\infty} f = \mp(1 - q^{\pm a}) \sum_{k=-\infty}^{\infty} (cq^{ak}) f(cq^{ak}),$$
$$(D_{q^{\pm a}})^{-1}\big|_{-\infty}^{0} f = \pm(1 - q^{\pm a}) \sum_{k=-\infty}^{\infty} (cq^{ak}) f(-cq^{ak}). \tag{189}$$

Notice that in (188) we introduced $\lambda_+ = q + q^{-1}$. The expressions in (185)-(188) allow us to read off the eigenvalues, i.e. the physical values which characterize the outcome of position (or momentum) measurements.

To reach this goal we assume the existence of a representation in which normally ordered monomials become diagonal [50]. In other words, the representation space is spanned by normalized vectors subject to

$$(X^1)^{i_1}(X^2)^{i_2} \ldots (X^n)^{i_n}|v^1, \ldots, v^n\rangle = (c_{v^1})^{i_1}(c_{v^2})^{i_2} \ldots (c_{v^n})^{i_n}|v^1, \ldots, v^n\rangle, \tag{190}$$

and

$$\langle u^1, \ldots, u^n|v^1, \ldots, v^n\rangle = \delta_{u^1, v^1} \ldots \delta_{u^n, v^n},$$
$$1 = \sum_{v^1, \ldots, v^n} |v^1, \ldots, v^n\rangle\langle v^1, \ldots, v^n|. \tag{191}$$

Each polynomial or power series of normally ordered monomials acts on this basis as

$$f(X^1, \ldots, X^n)|v^1, \ldots, v^n\rangle = f_{v^1, \ldots, v^n}|v^1, \ldots, v^n\rangle, \tag{192}$$

with

$$f_{v^1, \ldots, v^n} = \langle v^1, \ldots, v^n|f(X^1, \ldots, X^n)|v^1, \ldots, v^n\rangle,$$
$$= f(c_{v^1}, c_{v^2}, \ldots, c_{v^n}). \tag{193}$$

As soon as we know the values c_{v^i} we are done, since they determine the spectra of position operators. The polynomial or power series itself can be represented by the diagonal operator

$$f(X^1, \ldots, X^n) = \sum_{v^1, \ldots, v^n} |v^1, \ldots, v^n\rangle\, f_{v^1, \ldots, v^n}\, \langle v^1, \ldots, v^n|. \tag{194}$$

However, there is one subtlety we have to take care of. If we consider products of power series the coefficients in (193) have to be multiplied via

the star product, i.e.

$$
\begin{aligned}
& g(X^1,\ldots,X^n)f(X^1,\ldots,X^n) \\
& \quad = \sum_{v^1,\ldots,v^n} |v^1,\ldots,v^n\rangle\, g_{v^1,\ldots,v^n} \circledast f_{v^1,\ldots,v^n}\, \langle v^1,\ldots,v^n| \\
& \quad = \sum_{v^1,\ldots,v^n} |v^1,\ldots,v^n\rangle\, (g \circledast f)_{v^1,\ldots,v^n}\, \langle v^1,\ldots,v^n|.
\end{aligned} \tag{195}
$$

Finally, we introduce a vacuum state by

$$
|0\rangle \equiv \sum_{v^1,\ldots,v^n} |\, v^1,\ldots,v^n\rangle, \tag{196}
$$

for which we require to be invariant under translations and symmetry transformations, i.e.

$$
\partial^i|0\rangle = 0, \quad h|0\rangle = 0, \tag{197}
$$

where h denotes an element of a Hopf algebra describing the quantum symmetry of our quantum spaces.

In what follows it is important to realize that an integral over the whole space corresponds to an operator being diagonal in the following sense [47, 49, 50]:

$$
\int_{-\infty}^{+\infty} d_A^n x \; = \sum_{\underline{v}} |v^1,\ldots,v^n\rangle \Big(\int_{-\infty}^{+\infty} d_A^n x \Big)_{\underline{v},\underline{v}} \langle v^1,\ldots,v^n|. \tag{198}
$$

The integral of a function is then defined as an expectation value taken with respect to the vacuum state:

$$
\int_{-\infty}^{+\infty} d_A^n x\, f \equiv \langle 0| \int_{-\infty}^{+\infty} d_A^n x\, f(X^1,\ldots X^n)|0\rangle. \tag{199}
$$

Inserting (194), (196), and (198) into the right-hand side of the above equation leads us to

$$
\int_{-\infty}^{+\infty} d_A^n x\, f = \sum_{\underline{v}} \Big(\int_{-\infty}^{+\infty} d_A^n x \Big)_{\underline{v},\underline{v}} f_{\underline{v}}\,. \tag{200}
$$

Comparing this result with the expressions in (185)-(188), we can make the following identifications:

(i) (quantum plane)

$$\Big(\int_{-\infty}^{+\infty} d_L^2 x\Big)_{\underline{v},\underline{v}} = (q^2-1)^2(\pm\alpha_1 q^{2v^1})(\pm\alpha_2 q^{2v^2}),$$

$$f_{\underline{v}} = f(\pm\alpha_1 q^{2v^1}, \pm\alpha_2 q^{2v^2}), \quad v^i \in \mathbb{Z}, \tag{201}$$

(ii) (three-dimensional Euclidean space)

$$\Big(\int_{-\infty}^{+\infty} d_L^3 x\Big)_{\underline{v},\underline{v}} = (q^4-1)^2(q^2-1)(\pm\alpha_+ q^{4v^+})(\pm\alpha_3 q^{2v^3})(\pm\alpha_- q^{4v^-}),$$

$$f_{\underline{v}} = f(\pm\alpha_+ q^{4v^+}, \pm\alpha_3 q^{2v^3}, \pm\alpha_- q^{4v^-}), \quad v^A \in \mathbb{Z}, \tag{202}$$

(iii) (four-dimensional Euclidean space)

$$\Big(\int_{-\infty}^{+\infty} d_L^4 x\Big)_{\underline{v},\underline{v}} = (q^4-1)^4(\pm\alpha_1 q^{2v^1})(\pm\alpha_2 q^{2v^2})(\pm\alpha_3 q^{2v^3})(\pm\alpha_4 q^{2v^4}),$$

$$f_{\underline{v}} = f(\pm\alpha_1 q^{2v^1}, \pm\alpha_2 q^{2v^2}, \pm\alpha_3 q^{2v^3}, \pm\alpha_4 q^{2v^4}), \quad v^i \in \mathbb{Z}, \tag{203}$$

(iv) (q-deformed Minkowski space)

$$\Big(\int_{-\infty}^{+\infty} d_L^4 x\Big)_{\underline{v},\underline{v}} = (1-q^{-2})^4(\pm\alpha_{r^2} q^{2v^{r^2}})(\pm\alpha_+ q^{2v^+}) \times (\pm\alpha_{3/0} q^{2v^{3/0}})(\pm\alpha_- q^{2v^-}),$$

$$f_{\underline{v}} = f(\pm\alpha_{r^2} q^{2v^{r^2}}, \pm\alpha_+ q^{2v^+}, \pm\alpha_{3/0} q^{2v^{3/0}}, \pm\alpha_- q^{2v^-}), \quad v^\mu \in \mathbb{Z} \tag{204}$$

where the set of constants $\alpha_i \in \mathbb{C}$ characterizes the underlying representations (see also the discussion in the next section). This way, we found as matrix elements for our integral operator the q-deformed volume elements. (Notice that the size of these volume elements depends on their positions in space.) Furthermore, we see that the values for c_{v^i} are given by integer powers of q (up to a constant and a minus sign) This observation confirms the discrete character of our theory once more.

5 Physical interpretation of the formalism

In this section we would like to give the axioms of a q-deformed version of quantum kinematics . We start from the observables, which mathematically amount to operators on the vector spaces in question. The observables we are dealing with are position or momentum components. Opposed to the classical case the components X^i of the position operator do not commute among each other and the same holds for the components P^i of the momentum operator. More concretely, the components of position and momentum operator fulfill as commutation relations

$$(P_A)^{ij}_{kl}\, X^k X^l = 0, \quad (P_A)^{ij}_{kl}\, P^k P^l = 0. \tag{205}$$

where P_A denotes a q-analog of an antisymmetrizer. For this reason we cannot expect to find simultaneous eigenstates of the components of the position operator (momentum operator). In other words, there is no possibility to make sharp position (momentum) measurements. However, we can circumvent this problem if we require for the components of the position operator (momentum operator) to be applied in a certain order. This means, for example, that we first measure the component X^1, then the component X^2, and lastly the component X^n. Each measurement yields a certain value. Due to the non-commutativity of the components of the position (momentum) operator each measurement depends on the outcome of the previous ones. However, the incompatible measurements are carried out in a fixed order and this regulation guarantees that the physical states are uniquely characterized by the results for the measurements of the different components.

This interpretation is in complete accordance with the existence of a representation as it is defined in (190) and (191). The states for which normally ordered monomials in quantum space become diagonal are sometimes called *quasipoints* [50]. From the discussion in Sec. 4 we know that the set of quasipoints establishes a lattice whose spacings grow exponentially with the distance from the origin. The question now is how does the existence of discrete points fit together with the statement that we can not make sharp position measurements. This can be understood as follows. Each quasipoint represents a region having the size of the corresponding q-deformed volume element [cf. (201)-(204)]. If we change the order in which the components of the position operator are measured the coordinates for the quasipoint change, too. In other words, the coordinates describing the location of a quasipoint depend on the choice for the normal ordering of the components

of the position operator. However, all possible eigenvalues characterizing the same quasipoint for different normal orderings lead to ordinary points which lie in the region represented by the quasipoint under consideration. For this reason a quasipoint can be viewed as a region ('eigenregion') being labeled by the coordinates of a single point in that region. Normally, we have no control over the order in which the components of the position operator are applied when a measurement of position is performed. Thus, such a measurement can lead to any point within the region represented by a quasipoint, i.e. we only know in which 'eigenregion' our object is located, but we are unable to determine its precise position within this region.

Let us once more return to the commutation relations in (205). It should be obvious that they are part of q-deformed canonical commutation relations for position and momentum operators. It remains to write down the commutation relations between a momentum and a position operator. Since momentum operators amount to partial derivatives on position space, i.e. $P^k = \mathrm{i}\partial^k$, these relations follow immediately from the Leibniz rules for q-deformed partial derivatives. However, there are two choices for a differential calculus on quantum spaces. So we have two possibilities for a q-deformed commutator between momentum and position operators [32]:

$$\begin{aligned} P^k X^l - k(\hat{R}^{-1})^{kl}_{mn}\, X^m P^n &= \mathrm{i} g^{kl}, \\ P^k X^l - k^{-1} \hat{R}^{kl}_{mn}\, X^m P^n &= \mathrm{i} \bar{g}^{kl}, \end{aligned} \tag{206}$$

where $\hat{R}^{kl}_{mn}$ and $(\hat{R}^{-1})^{kl}_{mn}$ denote the vector representations of the universal R-matrix and its inverse, respectively. One should also notice that we introduced the quantum metric g^{kl} together with its conjugate version $\bar{g}^{ij}$. (In the case of the quantum plane it holds $\bar{g}^{ij} = -g^{ij}$ and otherwise we have $g^{ij} = \bar{g}^{ij}$.) For the concrete values of the constant k see Ref. [37].

Up to now, we have concentrated our attention on momentum and position operators. Next, we would like to say a few words about the vector spaces on which these operators act. As it was pointed out above momentum and position operators can both be represented on the space of quasipoints. Four our purposes it is sufficient to deal with a certain subspace of the vector space of quasipoints. The elements of this subspace are obtained as follows. Consider a polynomial or power series $f(X^i)$ $(f(P^i))$ in the components of the position operator (momentum operator). It is important that the components of position operator (momentum operator) are arranged in normal ordering. This task can always be achieved by the commutation relations in (205). Then we simply let $f(X^i)$ $(f(P^i))$ act on the vacuum state $|0\rangle$.

The subspace we are interested in consists of all elements we can get in this manner, i.e.

$$|\, f\,\rangle_x \equiv f(X^1,\ldots,X^n)|0\rangle_x, \tag{207}$$

and similarly for the momentum operator

$$|\, f\,\rangle_p \equiv f(P^1,\ldots,P^n)|0\rangle_p. \tag{208}$$

The state $|0\rangle_x$ is referred to as the vacuum of the *quasi-position representation* and $|0\rangle_p$ as that of the *quasi-momentum representation.* The two vacua are subject to

$$P^i|0\rangle_x = 0, \qquad X^i|0\rangle_p = 0, \tag{209}$$

i.e. $|0\rangle_x$ has zero momentum and $|0\rangle_p$ describes the origin of position space (where the observer is located). The corresponding dual spaces are made up by the vectors

$$\begin{aligned} {}_x\langle\, f\,| &\equiv {}_x\langle 0|\,\overline{f(x^1,\ldots,x^n)}, \\ {}_p\langle\, f\,| &\equiv {}_p\langle 0|\,\overline{f(p^1,\ldots,p^n)}. \end{aligned} \tag{210}$$

Finally, let us note that in this formalism the sesquilinear forms become

$$\begin{aligned} {}_x\langle\, f\,|\int_{-\infty}^{+\infty} d_A^n x\,|\, g\,\rangle_x &= \sum\nolimits_{\underline{v}}\Big(\int_{-\infty}^{+\infty} d_A^n x\Big)_{\underline{v},\underline{v}}(\bar{f} \overset{x}{\circledast} g)_{\underline{v}}, \\ {}_p\langle\, f\,|\int_{-\infty}^{+\infty} d_A^n p\,|\, g\,\rangle_p &= \sum\nolimits_{\underline{v}}\Big(\int_{-\infty}^{+\infty} d_A^n p\Big)_{\underline{v},\underline{v}}(\bar{f} \overset{p}{\circledast} g)_{\underline{v}}. \end{aligned} \tag{211}$$

Of course, nothing prevents us from introducing tensor products of the states in (207) and (208). Again, such states can be generated by applying functions, living in a tensor product of quantum spaces, to a tensor product of vacuum states. Such states become relevant if we are seeking solutions to the equations

$$X^i\,|\, f\,\rangle_{x\otimes\xi} = (X^i\otimes 1)|\, f\,\rangle_{x\otimes\xi} = |\, f \overset{\xi}{\circledast} \xi^i\,\rangle_{x\otimes\xi}, \tag{212}$$

and

$$P^k|\, f\,\rangle_{x\otimes p} = (\mathrm{i}\partial^k\otimes 1)|\, f\,\rangle_{x\otimes p} = |\, f \overset{p}{\circledast} p^k\,\rangle_{p\otimes x}. \tag{213}$$

From Sec. 3.1 we know that the states subject to these equations are generated by position and momentum eigenfunctions. Due to (51) and (55) we

have

$$
\begin{aligned}
X^i \,|\, u_y(x^j)\,\rangle_{x\otimes y} &= X^i\, u_Y(X^j)\,|0\rangle_{x\otimes y} \\
&= u_Y(X^j)\,(1\otimes Y^i)\,|0\rangle_{x\otimes y} \\
&= |\, u_y(x^j) \overset{y}{\circledast} y^i\,\rangle_{x\otimes y},
\end{aligned} \tag{214}
$$

and

$$
\begin{aligned}
\mathrm{i}\partial^k |\, u_p(x^j)\,\rangle_{x\otimes p} &= (\mathrm{i}\partial^k\otimes 1)\, u_P(X^j)\,|0\rangle_{x\otimes p} \\
&= u_P(X^j)\,(1\otimes P^k)\,|0\rangle_{x\otimes p} \\
&= |\, u_p(x^j) \overset{p}{\circledast} p^k\,\rangle_{x\otimes p}.
\end{aligned} \tag{215}
$$

To sum up, in the vector space of states $|\, f\,\rangle_{x\otimes y}$ there is a subspace spanned by the eigenstates of the position operator. Analogously, the vector space of states $|\, f\,\rangle_{x\otimes p}$ contains a subspace with momentum eigenstates. The states being dual to those in (214) and (215) can be respectively read off from the correspondences

$$
\begin{aligned}
{}_{x\otimes y}\langle\, (u_{\ominus_R y})_L(\ominus_L\, x^i)\,| &\leftrightarrow |\,(u_{\bar{R}})_y(x^i)\,\rangle_{x\otimes y}, \\
{}_{x\otimes y}\langle\, (u_{\ominus_L y})_R(\ominus_R\, x^i)\,| &\leftrightarrow |\,(u_{\bar{L}})_y(x^i)\,\rangle_{x\otimes y},
\end{aligned} \tag{216}
$$

and

$$
{}_{x\otimes p}\langle\, (\bar{u}_{R,\bar{L}})_{\ominus_R p}(x^i)\,| \leftrightarrow |\,(u_{R,\bar{L}})_p(x^i)\,\rangle_{x\otimes p}. \tag{217}
$$

These considerations also hold with some slight modifications if we use the eigenfunctions $\bar{u}_p(x^j)$ and $\bar{u}_\xi(x^j)$, instead.

From a physical point of view the subspace spanned by position (momentum) eigenfunctions is still too big. For this reason, let us return to the sesquilinear forms in (39) and (40). In analogy to the undeformed case we require for a physical wave function $\psi(x^k)$ in position space to satisfy

$$
\big\langle \psi(x^k), \psi(x^l)\big\rangle_{i,x} = 1, \tag{218}
$$

or alternatively

$$
\big\langle \psi(x^k), \psi(x^l)\big\rangle'_{i,x} = 1. \tag{219}
$$

This means that in the vector space spanned by position eigenfunctions we restrict attention to the subspace with elements fulfilling the property in (218) or (219) for a certain $i = 1, 2$. The elements of this subspace are referred to as *wave packet*s, since they can be written down as expansions

in terms of position or momentum eigenfunctions [cf. Sec. 3.2]. Obviously, (218) and (219) describe the postulate that physical wave functions have to be normalized to unity. This condition is essential for the probabilistic interpretation of quantum mechanics.

As is well-know position and momentum operators act on wave packets. Quantum mechanics tells us how to combine operators and wave-packets to yield measurable *expectation values*. We define the expectation value of an operator A taken with respect to a wave packet $\psi(x^k)$ by

$$\langle A_\psi \rangle_{i,x} \equiv \big\langle \psi, A \triangleright \psi \big\rangle_{i,x}, \tag{220}$$

or

$$\langle A_\psi \rangle'_{i,x} \equiv \big\langle \psi \triangleleft A, \psi \big\rangle'_{i,x}. \tag{221}$$

For this to make sense, we additionally require that

$$\overline{\langle A_\psi \rangle_{i,x}} = \langle A_\psi \rangle_{i,x}, \qquad \overline{\langle A_\psi \rangle'_{i,x}} = \langle A_\psi \rangle'_{i,x}. \tag{222}$$

This condition ensures that expectation values are real quantities. We will show that for real combinations of position or momentum operators these requirements are fulfilled:

$$\begin{aligned}
&\overline{\big\langle \tfrac{1}{2}(X^k + \overline{X^k})_\psi \big\rangle}_{i,x} \\
&\quad = \overline{\big\langle \psi, \tfrac{1}{2}(X^k + \overline{X^k}) \triangleright \psi \big\rangle}_{i,x} = \overline{\big\langle \psi, \tfrac{1}{2}(x^k + \overline{x^k}) \overset{x}{\circledast} \psi \big\rangle}_{i,x} \\
&\quad = \overline{\int_{-\infty}^{+\infty} d_i^n x \, \overline{\psi(x^j)} \overset{x}{\circledast} \tfrac{1}{2}(x^k + \overline{x^k}) \overset{x}{\circledast} \psi(x^l)} \\
&\quad = \int_{-\infty}^{+\infty} d_i^n x \, \overline{\psi(x^l)} \overset{x}{\circledast} \tfrac{1}{2}(x^k + \overline{x^k}) \overset{x}{\circledast} \psi(x^j) \\
&\quad = \big\langle \tfrac{1}{2}(X^k + \overline{X^k})_\psi \big\rangle_{i,x},
\end{aligned} \tag{223}$$

and

$$\begin{aligned}
&\overline{\big\langle \tfrac{1}{2}(P^k + \overline{P^k})_\psi \big\rangle}_{i,x} \\
&\quad = \overline{\big\langle \psi, \tfrac{1}{2}(P^k + \overline{P^k}) \triangleright \psi \big\rangle}_{i,x} = \overline{\big\langle \psi, \tfrac{1}{2}(\mathrm{i}\partial^k \triangleright \psi + \overline{\mathrm{i}\partial^k} \,\bar{\triangleright}\, \psi) \big\rangle}_{i,x}
\end{aligned}$$

$$
\begin{aligned}
&= \overline{\int_{-\infty}^{+\infty} d_i^n x\, \overline{\psi(x^j)} \overset{x}{\circledast} \frac{1}{2}(\mathrm{i}\partial^k \triangleright \psi(x^l) + \overline{\mathrm{i}\partial^k} \,\bar{\triangleright}\, \psi(x^l))} \\
&= \int_{-\infty}^{+\infty} d_i^n x\, \frac{1}{2}(\overline{\psi(x^l)} \triangleleft \mathrm{i}\partial^k + \overline{\psi(x^l)} \,\bar{\triangleleft}\, \overline{\mathrm{i}\partial^k}) \overset{x}{\circledast} \psi(x^j) \\
&= \int_{-\infty}^{+\infty} d_i^n x\, \overline{\psi(x^l)} \overset{x}{\circledast} \frac{1}{2}(\mathrm{i}\partial^k \triangleright \psi(x^j) + \overline{\mathrm{i}\partial^k} \,\bar{\triangleright}\, \psi(x^j)) \\
&= \big\langle \frac{1}{2}(P^k + \overline{P^k})_\psi \big\rangle_{i,x}.
\end{aligned} \tag{224}
$$

Similar arguments lead us to

$$
\begin{aligned}
\overline{\big\langle \frac{1}{2}(X^k + \overline{X^k})_\psi \big\rangle'_{i,x}} &= \big\langle \frac{1}{2}(X^k + \overline{X^k})_\psi \big\rangle'_{i,x}, \\
\overline{\big\langle \frac{1}{2}(P^k + \overline{P^k})_\psi \big\rangle'_{i,x}} &= \big\langle \frac{1}{2}(P^k + \overline{P^k})_\psi \big\rangle'_{i,x}.
\end{aligned} \tag{225}
$$

As we know from the discussion in Sec. 3.2 each wave function can be expanded in terms of position or momentum eigenfunctions. If a physical system is initially characterized by a wave function $\psi(x^i)$ the corresponding expansion coefficients should determine the probability for the system to be thrown into a certain quasipoint by a measurement of position or momentum. We also know that the expansion coefficients in a position basis are given by the wave function itself. Thus, it seems likely that the quantities

$$
(\rho)_\psi(x^k) \equiv \overline{\psi(x^k)} \overset{x}{\circledast} \psi(x^l), \tag{226}
$$

and

$$
(\rho)'_\psi(x^k) \equiv \psi(x^k) \overset{x}{\circledast} \overline{\psi(x^l)}, \tag{227}
$$

have the meaning of a probability density for finding a system at a certain position.

To get the probability density for finding a system in a certain momentum state, it is useful to rewrite the normalization conditions and the expectation values in terms of the Fourier coefficients of the wave function $\psi(x^i)$. It is well known that the Fourier coefficients determine the expansions of wave packets in terms of momentum eigenfunctions. Notice that we have different Fourier expansions for one and the same wave function $\psi(x^i)$. The Fourier coefficients being relevant for what follows are given by

$$
(\tilde{c}_A)_p \equiv \tilde{\mathcal{F}}_A(\psi(x^i))(p^k), \quad (\tilde{c}_A)^*_p \equiv \tilde{\mathcal{F}}^*_A(\psi(x^i))(p^k). \tag{228}
$$

Now, we are ready to rewrite the normalization conditions (218) and (219). With the help of the Fourier-Plancherel identities we get, at once,

$$\begin{aligned}
1 &= \big\langle \psi(x^k), \psi(x^l) \big\rangle_{1,x} \\
&= \frac{1}{2} \big\langle \tilde{\mathcal{F}}_{\bar{R}}(\psi), \tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i) \big\rangle_{1,p} + \frac{1}{2} \big\langle \tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i), \tilde{\mathcal{F}}_{\bar{R}}(\psi) \big\rangle_{1,p} \\
&= \int_{-\infty}^{+\infty} d_1 p \, \frac{1}{2} \big(\overline{(\tilde{c}_{\bar{R}})_p} \overset{p}{\circledast} (\tilde{c}^*_L)_{\kappa^{-1}p} + \overline{(\tilde{c}^*_L)_{\kappa^{-1}p}} \overset{p}{\circledast} (\tilde{c}_{\bar{R}})_p \big),
\end{aligned} \tag{229}$$

$$\begin{aligned}
1 &= \big\langle \psi(x^k), \psi(x^l) \big\rangle_{2,x} \\
&= \frac{1}{2} \big\langle \tilde{\mathcal{F}}_R(\psi), \tilde{\mathcal{F}}^*_{\bar{L}}(\psi)(\kappa p^i) \big\rangle_{1,p} + \frac{1}{2} \big\langle \tilde{\mathcal{F}}^*_{\bar{L}}(\psi)(\kappa p^i), \tilde{\mathcal{F}}_R(\psi) \big\rangle_{1,p} \\
&= \int_{-\infty}^{+\infty} d_2 p \, \frac{1}{2} \big(\overline{(\tilde{c}_R)_p} \overset{p}{\circledast} (\tilde{c}^*_{\bar{L}})_{\kappa p} + \overline{(\tilde{c}^*_{\bar{L}})_{\kappa p}} \overset{p}{\circledast} (\tilde{c}_R)_p \big),
\end{aligned} \tag{230}$$

and

$$\begin{aligned}
1 &= \big\langle \psi(x^k), \psi(x^l) \big\rangle'_{1,x} \\
&= \frac{1}{2} \big\langle \tilde{\mathcal{F}}_L(\psi), \tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i) \big\rangle'_{1,p} + \frac{1}{2} \big\langle \tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i), \tilde{\mathcal{F}}_L(\psi) \big\rangle'_{1,p} \\
&= \int_{-\infty}^{+\infty} d_1 p \, \frac{1}{2} \big((\tilde{c}_L)_p \overset{p}{\circledast} \overline{(\tilde{c}^*_{\bar{R}})_{\kappa^{-1}p}} + (\tilde{c}^*_{\bar{R}})_{\kappa^{-1}p} \overset{p}{\circledast} \overline{(\tilde{c}_L)_p} \big),
\end{aligned} \tag{231}$$

$$\begin{aligned}
1 &= \big\langle \psi(x^k), \psi(x^l) \big\rangle'_{2,x} \\
&= \frac{1}{2} \big\langle \tilde{\mathcal{F}}_{\bar{L}}(\psi), \tilde{\mathcal{F}}^*_R(\psi)(\kappa p^i) \big\rangle'_{2,p} + \frac{1}{2} \big\langle \tilde{\mathcal{F}}^*_R(\psi)(\kappa p^i), \tilde{\mathcal{F}}_{\bar{L}}(\psi) \big\rangle'_{2,p} \\
&= \int_{-\infty}^{+\infty} d_{\bar{L}/R}\, p \, \frac{1}{2} \big((\tilde{c}_{\bar{L}})_p \overset{p}{\circledast} \overline{(\tilde{c}^*_R)_{\kappa p}} + (\tilde{c}^*_R)_{\kappa p} \overset{p}{\circledast} \overline{(\tilde{c}_{\bar{L}})_p} \big).
\end{aligned} \tag{232}$$

It is also very instructive to formulate the expectation values of momentum operators in a momentum basis. Applying the Fourier-Plancherel identities now yields

$$\begin{aligned}
\big\langle \frac{1}{2}(P^k + \overline{P^k})_\psi \big\rangle_{1,x} &= \big\langle \psi, \frac{1}{2}(\mathrm{i}\partial^k \overset{x}{\triangleright} \psi + \overline{\mathrm{i}\partial^k} \overset{x}{\bar{\triangleright}} \psi) \big\rangle_{1,x} \\
&= \frac{1}{2} \big\langle \tilde{\mathcal{F}}_{\bar{R}}(\psi), \tilde{\mathcal{F}}^*_L(\mathrm{i}\partial^k \overset{x}{\triangleright} \psi)(\kappa^{-1}p^i) \big\rangle_{1,p} \\
&\quad + \frac{1}{2} \big\langle \tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i), \tilde{\mathcal{F}}_{\bar{R}}(\overline{\mathrm{i}\partial^k} \overset{x}{\bar{\triangleright}} \psi) \big\rangle_{1,p}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big\langle \tilde{\mathcal{F}}_{\bar{R}}(\psi), p^k \overset{p}{\circledast} \tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i)\big\rangle_{1,p} \\
&\quad + \frac{1}{2}\big\langle \tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i), \overline{p^k} \overset{p}{\circledast} \tilde{\mathcal{F}}_{\bar{R}}(\psi)\big\rangle_{1,p} \\
&= \int_{-\infty}^{+\infty} d_1\, p\, \frac{1}{2}\big(\,\overline{(\tilde{c}_{\bar{R}})_p} \overset{p}{\circledast} p^k \overset{p}{\circledast} (\tilde{c}^*_L)_{\kappa^{-1}p} \\
&\qquad\qquad + \overline{(\tilde{c}^*_L)_{\kappa^{-1}p}} \overset{p}{\circledast} \overline{p^k} \overset{p}{\circledast} (\tilde{c}_{\bar{R}})_p\big),
\end{aligned} \tag{233}
$$

and

$$
\begin{aligned}
\big\langle \frac{1}{2}(P^k + \overline{P^k})_\psi \big\rangle'_{1,x} &= \big\langle \frac{1}{2}(\psi \overset{x}{\triangleleft} (\mathrm{i}\partial^k) + \psi \overset{x}{\triangleleft} \overline{\mathrm{i}\partial^k}\,), \psi\big\rangle'_{1,x} \\
&= \frac{1}{2}\big\langle \tilde{\mathcal{F}}_L(\psi \overset{x}{\triangleleft} (\mathrm{i}\partial^k)), \tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i)\big\rangle'_{1,p} \\
&\quad + \frac{1}{2}\big\langle \tilde{\mathcal{F}}^*_{\bar{R}}(\psi \overset{x}{\triangleleft} \overline{\mathrm{i}\partial^k}\,)(\kappa^{-1}p^i), \tilde{\mathcal{F}}_L(\psi)\big\rangle'_{1,p} \\
&= \frac{1}{2}\big\langle \tilde{\mathcal{F}}_L(\psi) \overset{p}{\circledast} p^k, \tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i)\big\rangle'_{1,p} \\
&\quad + \frac{\kappa^n}{2}\big\langle \tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i) \overset{p}{\circledast} \overline{p^k}, \tilde{\mathcal{F}}_L(\psi)\big\rangle'_{1,x} \\
&= \int_{-\infty}^{+\infty} d_1\, p\, \frac{1}{2}\big((\tilde{c}_L)_p \overset{p}{\circledast} p^k \overset{p}{\circledast} \overline{(\tilde{c}^*_{\bar{R}})_{\kappa^{-1}p}} \\
&\qquad\qquad + (\tilde{c}^*_{\bar{R}})_{\kappa^{-1}p} \overset{p}{\circledast} \overline{p^k} \overset{p}{\circledast} \overline{(\tilde{c}_L)_p}\,\big).
\end{aligned} \tag{234}
$$

In very much the same we obtain

$$
\begin{aligned}
&\big\langle \frac{1}{2}(P^k + \overline{P^k})_\psi \big\rangle_{2,x} \\
&\quad = \int_{-\infty}^{+\infty} d_2\, p\, \frac{1}{2}\big(\,\overline{(\tilde{c}_R)_p} \overset{p}{\circledast} p^k \overset{p}{\circledast} (\tilde{c}^*_{\bar{L}})_{\kappa p} + \overline{(\tilde{c}^*_{\bar{L}})_{\kappa p}} \overset{p}{\circledast} \overline{p^k} \overset{p}{\circledast} (\tilde{c}_R)_p\big),
\end{aligned} \tag{235}
$$

$$
\begin{aligned}
&\big\langle \frac{1}{2}(P^k + \overline{P^k})_\psi \big\rangle'_{2,x} \\
&\quad = \int_{-\infty}^{+\infty} d_2\, p\, \frac{1}{2}\big((\tilde{c}_{\bar{L}})_p \overset{p}{\circledast} p^k \overset{p}{\circledast} \overline{(\tilde{c}^*_R)_{\kappa p}} + (\tilde{c}^*_R)_{\kappa p} \overset{p}{\circledast} \overline{p^k} \overset{p}{\circledast} \overline{(\tilde{c}_{\bar{L}})_p}\,\big).
\end{aligned} \tag{236}
$$

For the sake of completeness we wish to present the expectation value for position operators in a momentum basis. We have

$$
\big\langle \frac{1}{2}(X^k + \overline{X^k})_\psi \big\rangle_{1,x} = \big\langle \psi, \frac{1}{2}(x^k + \overline{x^k}) \overset{x}{\circledast} \psi\big\rangle_{1,x}
$$

$$= \frac{1}{2}\big\langle \tilde{\mathcal{F}}_{\bar{R}}(\psi), \tilde{\mathcal{F}}^*_L(\, x^k \overset{x}{\circledast} \psi)(\kappa^{-1}p^i)\big\rangle_{1,x}$$
$$+\ \frac{1}{2}\big\langle \tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i), \tilde{\mathcal{F}}_{\bar{R}}(\,\overline{x^k} \overset{x}{\circledast} \psi)\big\rangle_{1,x}$$
$$= \frac{1}{2}\big\langle \tilde{\mathcal{F}}_{\bar{R}}(\psi), \mathrm{i}\partial^k \overset{p}{\triangleright} \tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i)\big\rangle_{1,x}$$
$$+\ \frac{1}{2}\big\langle \tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i), \overline{\mathrm{i}\partial^k} \overset{p}{\triangleright} \tilde{\mathcal{F}}_{\bar{R}}(\psi)\big\rangle_{1,x}$$
$$= \int_{-\infty}^{+\infty} d_1\, p\, \frac{1}{2}\big(\,\overline{(\tilde{c}_{\bar{R}})_p} \overset{p}{\circledast} \big(\mathrm{i}\partial^k \overset{p}{\triangleright} (\tilde{c}^*_L)_{\kappa^{-1}p}\big)$$
$$+\ \overline{(\tilde{c}^*_L)_{\kappa^{-1}p}} \overset{p}{\circledast} \big(\,\overline{\mathrm{i}\partial^k} \overset{p}{\triangleright} (\tilde{c}_{\bar{R}})_p\big)\big), \tag{237}$$

and

$$\big\langle \frac{1}{2}(X^k + \overline{X^k})_\psi \big\rangle'_{1,x} = \big\langle \psi \overset{x}{\circledast} \frac{1}{2}(x^k + \overline{x^k}), \psi\big\rangle'_{1,x}$$
$$= \frac{1}{2}\big\langle \tilde{\mathcal{F}}_L(\psi \overset{x}{\circledast} x^k), \tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i)\big\rangle_{1,x}$$
$$+\ \frac{1}{2}\big\langle \tilde{\mathcal{F}}^*_{\bar{R}}(\psi \overset{x}{\circledast} \overline{x^k}\,)(\kappa^{-1}p^i), \tilde{\mathcal{F}}_L(\psi)\big\rangle'_{1,x}$$
$$= \frac{1}{2}\big\langle \tilde{\mathcal{F}}_L(\psi) \overset{p}{\triangleleft} (\mathrm{i}\partial^k), \tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i)\big\rangle_{1,x}$$
$$+\ \frac{1}{2}\big\langle \tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i) \overset{p}{\triangleleft} \overline{\mathrm{i}\partial^k}, \tilde{\mathcal{F}}_L(\psi)\big\rangle'_{1,x}$$
$$= \int_{-\infty}^{+\infty} d_{L/\bar{R}}\, p\, \frac{1}{2}\big(\big((\tilde{c}_L)_p \overset{p}{\triangleleft} (\mathrm{i}\partial^k)\big) \overset{p}{\circledast} \overline{(\tilde{c}^*_{\bar{R}})_{\kappa^{-1}p}}$$
$$+\ \big((\tilde{c}^*_{\bar{R}})_{\kappa^{-1}p} \overset{p}{\triangleleft} \overline{\mathrm{i}\partial^k}\,\big) \overset{p}{\circledast} \overline{(\tilde{c}_L)_p}\,\big). \tag{238}$$

Likewise we get

$$\big\langle \frac{1}{2}(X^k + \overline{X^k})_\psi \big\rangle_{2,x} = \int_{-\infty}^{+\infty} d_2\, p\, \frac{1}{2}\big(\,\overline{(\tilde{c}_R)_p} \overset{p}{\circledast} \big(\mathrm{i}\hat{\partial}^k \overset{p}{\triangleright} (\tilde{c}^*_{\bar{L}})_{\kappa p}\big)$$
$$+\ \overline{(\tilde{c}^*_{\bar{L}})_{\kappa p}} \overset{p}{\circledast} \big(\,\overline{\mathrm{i}\hat{\partial}^k} \overset{p}{\triangleright} (\tilde{c}_R)_p\big)\big), \tag{239}$$

$$\big\langle \frac{1}{2}(X^k + \overline{X^k})_\psi \big\rangle'_{2,x} = \int_{-\infty}^{+\infty} d_2\, p\, \frac{1}{2}\big(\big((\tilde{c}_{\bar{L}})_p \overset{p}{\triangleleft} (\mathrm{i}\hat{\partial}^k)\big) \overset{p}{\circledast} \overline{(\tilde{c}^*_R)_{\kappa p}}$$
$$+\ \big((\tilde{c}^*_R)_{\kappa p} \overset{p}{\triangleleft} \overline{\mathrm{i}\hat{\partial}^k}\,\big) \overset{p}{\circledast} \overline{(\tilde{c}_{\bar{L}})_p}\,\big). \tag{240}$$

These formulae should make it obvious that the probability densities for

meeting a system in a quasi-momentum state are given by

$$
\begin{aligned}
(\rho_1)_\psi(p^k) &\equiv \frac{1}{2}\big(\,\overline{\tilde{\mathcal{F}}_{\bar{R}}(\psi)} \overset{p}{\circledast} \tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i) + \overline{\tilde{\mathcal{F}}^*_L(\psi)(\kappa^{-1}p^i)} \overset{p}{\circledast} \tilde{\mathcal{F}}_{\bar{R}}(\psi)\big) \\
&= \frac{1}{2}\big(\,\overline{(\tilde{c}_{\bar{R}})_p} \overset{p}{\circledast} (\tilde{c}^*_L)_{\kappa^{-1}p} + \overline{(\tilde{c}^*_L)_{\kappa^{-1}p}} \overset{p}{\circledast} (\tilde{c}_{\bar{R}})_p\big),
\end{aligned} \tag{241}
$$

$$
\begin{aligned}
(\rho_2)_\psi(p^k) &\equiv \frac{1}{2}\big(\,\overline{\tilde{\mathcal{F}}_R(\psi)} \overset{p}{\circledast} \tilde{\mathcal{F}}^*_{\bar{L}}(\psi)(\kappa p^i) + \overline{\tilde{\mathcal{F}}^*_{\bar{L}}(\psi)(\kappa p^i)} \overset{p}{\circledast} \tilde{\mathcal{F}}_R(\psi)\big) \\
&= \frac{1}{2}\big(\,\overline{(\tilde{c}_R)_p} \overset{p}{\circledast} (\tilde{c}^*_{\bar{L}})_{\kappa p} + \overline{(\tilde{c}^*_{\bar{L}})_{\kappa p}} \overset{p}{\circledast} (\tilde{c}_R)_p\big),
\end{aligned} \tag{242}
$$

and

$$
\begin{aligned}
(\rho_1')_\psi(p^k) &\equiv \frac{1}{2}\big(\tilde{\mathcal{F}}_L(\psi) \overset{p}{\circledast} \overline{\tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i)} + \tilde{\mathcal{F}}^*_{\bar{R}}(\psi)(\kappa^{-1}p^i) \overset{p}{\circledast} \overline{\tilde{\mathcal{F}}_L(\psi)}\,\big) \\
&= \frac{1}{2}\big((\tilde{c}_L)_p \overset{p}{\circledast} \overline{(\tilde{c}^*_{\bar{R}})_{\kappa^{-1}p}} + (\tilde{c}^*_{\bar{R}})_{\kappa^{-1}p} \overset{p}{\circledast} \overline{(\tilde{c}_L)_p}\,\big),
\end{aligned} \tag{243}
$$

$$
\begin{aligned}
(\rho_2')_\psi(p^k) &\equiv \frac{1}{2}\big(\tilde{\mathcal{F}}_{\bar{L}}(\psi) \overset{p}{\circledast} \overline{\tilde{\mathcal{F}}^*_R(\psi)(\kappa p^i)} + \tilde{\mathcal{F}}^*_R(\psi)(\kappa p^i) \overset{p}{\circledast} \overline{\tilde{\mathcal{F}}_{\bar{L}}(\psi)}\,\big) \\
&= \frac{1}{2}\big((\tilde{c}_{\bar{L}})_p \overset{p}{\circledast} \overline{(\tilde{c}^*_R)_{\kappa p}} + (\tilde{c}^*_R)_{\kappa p} \overset{p}{\circledast} \overline{(\tilde{c}_{\bar{L}})_p}\,\big).
\end{aligned} \tag{244}
$$

6 Quantum mechanics with Grassmann variables

Up to know attention was focused on symmetrized quantum spaces, only. However, nothing prevents us from considering antisymmetrized quantum spaces as well. In Refs. [24] and [25] it was shown that all arguments leading to q-analysis on symmetrized quantum spaces carry over to antisymmetrized quantum spaces without any difficulties. The only difference is that for q-deformed superanalysis we have to deal with the category of antisymmetrized quantum spaces, while the objects of q-analysis refer to the category of symmetrized quantum spaces.

In this manner, it should be evident that the considerations in Part I of this paper, especially those about Fourier transformations, remain valid if we substitute for the operations of q-analysis the corresponding ones of q-deformed superanalysis. More concretely, this means that star products have to be replaced by the q-deformed Grassmann product:

$$
f(x^i) \overset{x}{\circledast} g(x^j) \rightarrow f(\theta^i) \overset{\theta}{\cdot} g(\theta^j). \tag{245}
$$

Furthermore, we have to use braided products and q-translations for antisymmetrized quantum spaces. Instead of partial derivatives acting on symmetrized quantum spaces we now apply derivatives for q-deformed Grassmann variables:

$$\begin{aligned}\partial^i \overset{x}{\triangleright} f(x^j) &\rightarrow \partial^i \overset{\theta}{\triangleright} f(\theta^j),\\ \hat{\partial}^i \overset{x}{\bar{\triangleright}} f(x^j) &\rightarrow \hat{\partial}^i \overset{\theta}{\bar{\triangleright}} f(\theta^j),\end{aligned}\tag{246}$$

$$\begin{aligned}f(x^j) \overset{x}{\triangleleft} \hat{\partial}^i &\rightarrow f(\theta^j) \overset{\theta}{\triangleleft} \hat{\partial}^i,\\ f(x^j) \overset{x}{\bar{\triangleleft}} \partial^i &\rightarrow f(\theta^j) \overset{\theta}{\bar{\triangleleft}} \partial^i.\end{aligned}\tag{247}$$

Finally, integrals and exponentials on symmetrized quantum spaces have to be substituted by their counterparts for antisymmetrized quantum spaces:

$$\int_{-\infty}^{+\infty} d_A^n x\, f(x^i) \rightarrow \int d_A^n \theta\, f(\theta^i),\tag{248}$$

$$\begin{aligned}\exp(x^k|\mathrm{i}^{-1}p^l)_{A,B} &\rightarrow \exp(\theta^k|\mathrm{i}^{-1}\rho^l)_{A,B},\\ \exp(\mathrm{i}^{-1}p^l|x^k)_{A,B} &\rightarrow \exp(\mathrm{i}^{-1}\rho^l|\theta^k)_{A,B}.\end{aligned}\tag{249}$$

We are now ready to write down Fourier transformations for Grassmann variables, i.e.

$$\begin{aligned}\mathcal{F}_L(f)(\rho^k) &\equiv \int d_L^n\theta\, f(\theta^i) \overset{\theta}{\cdot} \exp(\theta^j|\mathrm{i}^{-1}\rho^k)_{\bar{R},L},\\ \mathcal{F}_{\bar{L}}(f)(\rho^k) &\equiv \int d_{\bar{L}}^n x\, f(\theta^i) \overset{\theta}{\cdot} \exp(\theta^j|\mathrm{i}^{-1}\rho^k)_{R,\bar{L}},\end{aligned}\tag{250}$$

$$\begin{aligned}\mathcal{F}_R(f)(\rho^k) &\equiv \int d_R^n x\, \exp(\mathrm{i}^{-1}\rho^k|\theta^j)_{R,\bar{L}} \overset{\theta}{\cdot} f(\theta^i),\\ \mathcal{F}_{\bar{R}}(f)(\rho^k) &\equiv \int d_{\bar{R}}^n x\, \exp(\mathrm{i}^{-1}\rho^k|\theta^j)_{\bar{R},L} \overset{\theta}{\cdot} f(\theta^i).\end{aligned}\tag{251}$$

Notice that the Grassmann variables ρ^k play the role of momentum coordinates on antisymmetrized quantum spaces. In complete analogy to the symmetrized quantum spaces, delta functions for Grassmann variables are

given by

$$\delta^n_L(\rho^k) \equiv \mathcal{F}_L(1)(\rho^k) = \int d^n_L\theta \exp(\theta^j|\mathrm{i}^{-1}\rho^k)_{\bar{R},L},$$
$$\delta^n_{\bar{L}}(\rho^k) \equiv \mathcal{F}_{\bar{L}}(1)(\rho^k) = \int d^n_{\bar{L}}\theta \exp(\theta^j|\mathrm{i}^{-1}\rho^k)_{R,\bar{L}}, \tag{252}$$

$$\delta^n_R(\rho^k) \equiv \mathcal{F}_R(1)(\rho^k) = \int d^n_R\theta \ \exp(\mathrm{i}^{-1}\rho^k|\theta^j)_{R,\bar{L}},$$
$$\delta^n_{\bar{R}}(\rho^k) \equiv \mathcal{F}_{\bar{R}}(1)(\rho^k) = \int d^n_{\bar{R}}\theta \ \exp(\mathrm{i}^{-1}\rho^k|\theta^j)_{\bar{R},L}. \tag{253}$$

Inserting the concrete expressions for integrals and exponentials (see Ref. [25]) a straightforward calculation yields

(i) (quantum plane)

$$\delta^2_L(\rho^k) = \delta^2_{\bar{R}}(\rho^k) = \rho^2\rho^1,$$
$$\delta^2_{\bar{L}}(\rho^k) = \delta^2_R(\rho^k) = \rho^1\rho^2, \tag{254}$$

(ii) (three-dimensional Euclidean space)

$$\delta^3_L(\rho^k) = \delta^3_{\bar{R}}(\rho^k) = \mathrm{i}\,\rho^+\rho^3\rho^-,$$
$$\delta^3_{\bar{L}}(\rho^k) = \delta^3_R(\rho^k) = \mathrm{i}\,\rho^-\rho^3\rho^+, \tag{255}$$

(iii) (four-dimensional Euclidean space)

$$\delta^4_L(\rho^k) = \delta^4_{\bar{R}}(\rho^k) = \rho^4\rho^3\rho^2\rho^1,$$
$$\delta^4_{\bar{L}}(\rho^k) = \delta^4_R(\rho^k) = \rho^1\rho^2\rho^3\rho^4, \tag{256}$$

(iv) (q-deformed Minkowski space)

$$\delta^4_L(\rho^k) = \rho^-\rho^{3/0}\rho^3\rho^+, \quad \delta^4_R(\rho^k) = \rho^+\rho^3\rho^{3/0}\rho^-,$$
$$\delta^4_{\bar{L}}(\rho^k) = \rho^+\rho^{3/0}\rho^3\rho^-, \quad \delta^4_{\bar{R}}(\rho^k) = \rho^-\rho^3\rho^{3/0}\rho^+. \tag{257}$$

The volume elements for antisymmetrized quantum spaces are again defined

as integrals of delta functions:

$$\begin{aligned}
\mathrm{vol}_L &\equiv \int d^n_{\bar{R}}\rho\, \delta^n_L(\rho^k), \\
\mathrm{vol}_{\bar{L}} &\equiv \int d^n_R\rho\, \delta^n_{\bar{L}}(\rho^k),
\end{aligned} \tag{258}$$

$$\begin{aligned}
\mathrm{vol}_R &\equiv \int d^n_{\bar{L}}\rho\, \delta^n_R(\rho^k), \\
\mathrm{vol}_{\bar{R}} &\equiv \int d^n_L\rho\, \delta^n_{\bar{R}}(\rho^k).
\end{aligned} \tag{259}$$

From these definitions together with the results of Ref. [25] we get, at once,

(i) (quantum plane) $\mathrm{vol}_A = 1$,

(ii) (three-dimensional Euclidean space) $\mathrm{vol}_A = \mathrm{i}$,

(iii) (four-dimensional Euclidean space) $\mathrm{vol}_A = 1$,

(iv) (q-deformed Minkowski space) $\mathrm{vol}_A = 1$,

where $A \in \{L, \bar{L}, R, \bar{R}\}$.

All relations fulfilled by the elements of q-analysis have a counterpart in q-deformed superanalysis. The reason for this lies in the fact that q-deformed superanalysis is based on the same abstract ideas as q-analysis. Thus, the substitutions in (245)-(249) represent an easy way to derive the identities of q-deformed superanalysis from those of q-analysis.

Next, we turn to sesquilinear forms for supernumbers. They are given by

$$\begin{aligned}
\big\langle f, g \big\rangle_{A,\theta} &\equiv \int d^n_A\theta\, \overline{f(\theta^i)} \overset{\theta}{\cdot} g(\theta^j), \\
\big\langle f, g \big\rangle'_{A,\theta} &\equiv \int d^n_A\theta\, f(\theta^i) \overset{\theta}{\cdot} \overline{g(\theta^j)}.
\end{aligned} \tag{260}$$

To be prepared for concrete calculations we would like to present explicit formulae for these sesquilinear forms. (I am very grateful to Alexander Schmidt for doing these calculations with Mathematica.) Using the results and the notation of Refs. [24, 25] we have

(i) (quantum plane)

$$\big\langle f, g \big\rangle_{L,\theta} = \big\langle f, g \big\rangle_{\bar{R},\theta}$$

$$= \overline{f'}\, g_{21} - \overline{f_{21}}\, g' + q^{-1/2}\overline{f_1}\, g_1 + q^{-1/2}\overline{f_2}\, g_2, \tag{261}$$

$$\begin{aligned} \big\langle f,g\big\rangle_{\bar{L},\theta} &= \big\langle f,g\big\rangle_{R,\theta} \\ &= \overline{f'}\, g_{12} - \overline{f_{12}}\, g' - q^{1/2}\overline{f_1}\, g_1 - q^{1/2}\overline{f_2}\, g_2, \end{aligned} \tag{262}$$

$$\begin{aligned} \big\langle f,g\big\rangle'_{L,\theta} &= \big\langle f,g\big\rangle'_{\bar{R},\theta} \\ &= -f'\,\overline{g_{21}} + f_{21}\,\overline{g'} - q^{-3/2} f_1\,\overline{g_1} - q^{1/2} f_2\,\overline{g_2}, \end{aligned} \tag{263}$$

$$\begin{aligned} \big\langle f,g\big\rangle'_{\bar{L},\theta} &= \big\langle f,g\big\rangle'_{R,\theta} \\ &= -f'\,\overline{g_{12}} + f_{12}\,\overline{g'} + q^{-1/2} f_1\,\overline{g_1} + q^{3/2} f_2\,\overline{g_2}, \end{aligned} \tag{264}$$

(ii) (three-dimensional Euclidean space)

$$\begin{aligned} \big\langle f,g\big\rangle_{L,\theta} &= \big\langle f,g\big\rangle_{\bar{R},\theta} \\ &= \overline{f'}\, g_{+3-} - q^{-1}\overline{f_+}\, g_{+3} - q^{-2}\overline{f_3}\, g_{+-} - q^{-1}\overline{f_-} g_{3-} \\ &\quad + q^{-3}\overline{f_{+3}} g_+ + q^{-2}\overline{f_{+-}} g_3 + q^{-3}\overline{f_{3-}} g_- - q^{-4}\overline{f_{+3-}} g', \end{aligned} \tag{265}$$

$$\begin{aligned} \big\langle f,g\big\rangle_{\bar{L},\theta} &= \big\langle f,g\big\rangle_{R,\theta} \\ &= \overline{f'} g_{-3+} - q\overline{f_-} g_{-3} - q^2\overline{f_3} g_{-+} - q\overline{f_+} g_{3+} \\ &\quad + q^3\overline{f_{-3}} g_- + q^2\overline{f_{-+}} g_3 + q^3\overline{f_{3+}} g_+ - q^4\overline{f_{-3+}} g', \end{aligned} \tag{266}$$

$$\begin{aligned} \big\langle f,g\big\rangle'_{L,\theta} &= \big\langle f,g\big\rangle'_{\bar{R},\theta} \\ &= -q^{-4} f'\overline{g_{+3-}} + q^{-1} f_+\overline{g_{+3}} + q^{-2} f_3\overline{g_{+-}} + q^{-5} f_-\overline{g_{3-}} \\ &\quad - q f_{3+}\overline{g_+} - q^{-2} f_{+-}\overline{g_3} - q^{-3} f_{3-}\overline{g_-} + f_{+3-}\overline{g'}, \end{aligned} \tag{267}$$

$$\begin{aligned} \big\langle f,g\big\rangle'_{\bar{L},\theta} &= \big\langle f,g\big\rangle'_{R,\theta} \\ &= -q^4 f'\overline{g_{-3+}} + q^{-1} f_-\overline{g_{3-}} + q^2 f_3\overline{g_{-+}} + q^5 f_+\overline{g_{3+}}, \\ &\quad - q^{-1} f_{-3}\overline{g_-} - q^2 f_{-+}\overline{g_3} - q^3 f_{3+}\overline{g_+} + f_{-3+}\overline{g'}, \end{aligned} \tag{268}$$

(iii) (four-dimensional Euclidean space)

$$\begin{aligned} \big\langle f,g\big\rangle_{L,\theta} &= \big\langle f,g\big\rangle_{\bar{R},\theta} \\ &= \overline{f'}\, g_{1234} - q\overline{f_1}\, g_{123} + q\overline{f_2}\, g_{124} - q\overline{f_3}\, g_{134} \\ &\quad - q^2\overline{f_{12}}\, g_{12} + q^2\overline{f_{13}}\, g_{13} - q^2\overline{f_{23}}\, g_{14} - q^2\overline{f_{14}}\, g_{23} \\ &\quad + q^2\overline{f_{24}}\, g_{24} - q^2\overline{f_{34}}\, g_{34} + q^3\overline{f_{123}}\, g_1 - q^3\overline{f_{124}}\, g_2 \end{aligned}$$

$$+\; q^3\overline{f_{134}}\,g_3 + q^4\overline{f_{1234}}\,g', \tag{269}$$

$$\begin{aligned}
\big\langle f,g\big\rangle_{\bar{L},\theta} &= \big\langle f,g\big\rangle_{R,\theta} \\
&= \overline{f'}\,g_{4321} + q^{-1}\overline{f_3}\,g_{431} - q^{-1}\overline{f_2}\,g_{421} + q^{-1}\overline{f_1}\,g_{321} \\
&\quad -\; q^{-2}\overline{f_{43}}g_{43} + q^{-2}\overline{f_{42}}g_{42} - q^{-2}\overline{f_{32}}\,g_{41} - q^{-2}\overline{f_{41}}\,g_{32} \\
&\quad +\; q^{-2}\overline{f_{31}}\,g_{31} - q^{-2}\overline{f_{21}}\,g_{21} + q^{-3}\overline{f_{321}}\,g_1 - q^{-3}\overline{f_{431}}\,g_3 \\
&\quad +\; q^{-3}\overline{f_{421}}\,g_2 + q^{-4}\overline{f_{4321}}\,g',
\end{aligned} \tag{270}$$

$$\begin{aligned}
\big\langle f,g\big\rangle'_{L,\theta} &= \big\langle f,g\big\rangle'_{\bar{R},\theta} \\
&= q^4 f'\,\overline{g_{1234}} - qf_1\,\overline{g_{123}} + q^3 f_2\,\overline{g_{124}} - q^3 f_3\,\overline{g_{134}} \\
&\quad -\; f_{12}\,\overline{g_{12}} + f_{13}\,\overline{g_{13}} - q^2 f_{23}\,\overline{g_{14}} + q^4 f_{24}\,\overline{g_{24}} \\
&\quad -\; q^4 f_{34}\,\overline{g_{34}} - q^2 f_{14}\,\overline{g_{23}} + q^{-1} f_{123}\,\overline{g_1} - qf_{124}\,\overline{g_2} \\
&\quad +\; qf_{134}\,\overline{g_3} + f_{1234}\,\overline{g'},
\end{aligned} \tag{271}$$

$$\begin{aligned}
\big\langle f,g\big\rangle'_{\bar{L},\theta} &= \big\langle f,g\big\rangle'_{R,\theta} \\
&= q^{-4} f'\,\overline{g_{4321}} + q^{-3} f_3\,\overline{g_{431}} - q^{-3} f_2\,\overline{g_{421}} + q^{-5} f_1\,\overline{g_{321}}, \\
&\quad -\; f_{43}\,\overline{g_{43}} + f_{24}\,\overline{g_{24}} - q^{-2} f_{32}\,\overline{g_{41}} + q^{-4} f_{31}\,\overline{g_{31}} \\
&\quad -\; q^{-4} f_{21}\,\overline{g_{21}} - q^{-2} f_{41}\,\overline{g_{32}} - q^{-1} f_{431}\,\overline{g_3} + q^{-1} f_{421}\,\overline{g_2} \\
&\quad -\; q^{-3} f_{321}\,\overline{g_1} + f_{4321}\,\overline{g'},
\end{aligned} \tag{272}$$

(iv) (q-deformed Minkowski space)

$$\begin{aligned}
&\big\langle f,g\big\rangle_{L,\theta} \\
&\quad = \overline{f'}\,g_{-,3/0,3+} + q\overline{f_-}\,g_{3/0,3-} - \overline{f_{3/0}}\,g_{-3+} + q^2\overline{f_3}\,g_{-,3/0,+} \\
&\quad -\; q\overline{f_+}\,g_{3/0,3,+} + q^3\overline{f_{-3}}\,g_{-,3/0} - q\overline{f_{-,3/0}}\,g_{-3} - q^2\overline{f_{-+}}\,g_{3/0,3} \\
&\quad +\; q^2\overline{f_{3/0,3}}\,g_{-+} + (q-q^3)\,\overline{f_{3/0,3}}\,g_{3/0,3} - q^3\overline{f_{3+}}\,g_{3/0,+} \\
&\quad +\; q\overline{f_{3/0,+}}\,g_{3+} + q^3\overline{f_{-,3/0,3}}\,g_- - q^4\overline{f_{-3+}}\,g_{3/0} + q^2\overline{f_{-,3/0,+}}\,g_3 \\
&\quad -\; q^3\overline{f_{3/0,3+}}\,g_+ - q^4\overline{f_{-,3/0,3+}}\,g',
\end{aligned} \tag{273}$$

$$\begin{aligned}
&\big\langle f,g\big\rangle_{\bar{L},\theta} \\
&\quad = \overline{f'}\,g_{+3,3/0,-} + q^{-1}\overline{f_+}\,g_{3/0,3} - \overline{f_{3/0}}\,g_{+3-} + q^{-2}\overline{f_3}\,g_{+,3/0,-} \\
&\quad -\; q^{-1}\overline{f_-}\,g_{3/0,3,-} + q^{-3}\overline{f_{+3}}\,g_{+,3/0} - q^{-1}\overline{f_{+,3/0}}\,g_{+3} \\
&\quad -\; q^{-2}\overline{f_{+-}}\,g_{3/0,3} + q^{-2}\overline{f_{3/0,3}}\,g_{+-} + (q^{-1}-q^{-3})\,\overline{f_{3/0,3}}g_{3/0,3}
\end{aligned}$$

$$
\begin{aligned}
&- q^{-3}\overline{f_{3-}}\, g_{3/0,-} + q^{-1}\overline{f_{3/0,-}}\, g_{3-} + q^{-3}\overline{f_{+,3/0,3}}\, g_{+} \\
&- q^{-4}\overline{f_{+3-}}\, g_{3/0} + q^{-2}\overline{f_{+,3/0,-}}\, g_{3} - q^{-3}\overline{f_{3/0,3-}}\, g_{-} \\
&- q^{-4}\overline{f_{+,3/0,3-}}\, g', \qquad (274)
\end{aligned}
$$

$$
\begin{aligned}
\big\langle f,g\big\rangle_{R,\theta} \\
= {}& \overline{f'}\, g_{+3,3/0,-} + q^{-1}\overline{f_{+}}\, g_{+3,3/0} - q^{-2}\overline{f_{3}}\, g_{+,3/0,-} + \overline{f_{3/0}}\, g_{+3-} \\
&- q^{-1}\overline{f_{-}}\, g_{3,3/0,-} - q^{-3}\overline{f_{+3}}\, g_{+,3/0} + q^{-1}\overline{f_{+,3/0}}\, g_{+3} \\
&- q^{-2}\overline{f_{+-}}\, g_{3,3/0} + q^{-2}\overline{f_{3,3/0}}\, g_{+-} + (q^{-1} - q^{-3})\,\overline{f_{3,3/0}}\, g_{3,3/0} \\
&+ q^{-3}\overline{f_{3-}}\, g_{3/0,-} - q^{-1}\overline{f_{3/0,-}}\, g_{3-} + q^{-3}\overline{f_{+3,3/0}}\, g_{+} \\
&+ q^{-4}\overline{f_{+3-}}\, g_{3/0} - q^{-2}\overline{f_{+,3/0,-}}\, g_{3} - q^{-3}\overline{f_{3,3/0,-}}\, g_{-} \\
&- q^{-4}\overline{f_{+3,3/0,-}}\, g', \qquad (275)
\end{aligned}
$$

$$
\begin{aligned}
\big\langle f,g\big\rangle_{\bar{R},\theta} \\
= {}& \overline{f'}g_{-3,3/0,+} + q\overline{f_{-}}g_{-3,3/0} - q^{2}\overline{f_{3}}g_{+,3/0,-} + \overline{f_{3/0}}g_{-3+} \\
&- q\overline{f_{+}}g_{3,3/0,+} - q^{3}\overline{f_{-3}}g_{-,3/0} + q\overline{f_{-,3/0}}g_{-3} \\
&- q^{2}\overline{f_{-+}}g_{3,3/0} + q^{2}\overline{f_{3,3/0}}g_{-+} + (q - q^{3})\overline{f_{3,3/0}}g_{3,3/0} \\
&+ q^{3}\overline{f_{3+}}g_{3/0,+} - q\overline{f_{3/0,+}}g_{3+} + q^{3}\overline{f_{-3,3/0}}g_{-} + q^{4}\overline{f_{-3+}}g_{3/0} \\
&- q^{2}\overline{f_{-,3/0,+}}g_{3} - q^{3}\overline{f_{3,3/0,+}}g_{+} - q^{4}\overline{f_{-3,3/0,+f}}g', \qquad (276)
\end{aligned}
$$

and

$$
\begin{aligned}
\big\langle f,g\big\rangle'_{L,\theta} \\
= {}& -q^{4}f'\overline{g_{-,3/0,3+}} - qf_{-}\overline{g_{-,3/0,3}} + q^{4}f_{3/0}\overline{g_{-3+}} - q^{2}f_{3}\overline{g_{-,3/0,+}} \\
&+ q^{5}f_{+}\overline{g_{3/0,3+}} - q^{-1}f_{-3}\overline{g_{-,3/0}} + qf_{-,3/0}\overline{g_{-3}} + q^{2}f_{-+}\overline{g_{3/0,3}} \\
&+ (q - q^{3})f_{3/0,3}\overline{g_{3/0,3}} - q^{2}f_{3/0,3}\overline{g_{-+}} - q^{5}f_{3/0,+}\overline{g_{3+}} \\
&+ q^{3}f_{3+}\overline{g_{3/0,+}} - q^{-1}f_{-,3/0,3}\overline{g_{-}} - q^{2}f_{-,3/0,+}\overline{g_{3}} + f_{-3+}\overline{g_{3/0}} \\
&+ q^{3}f_{3/0,3+}\overline{g_{+}} + f_{-,3/0,3+}\overline{g'}, \qquad (277)
\end{aligned}
$$

$$
\begin{aligned}
\big\langle f,g\big\rangle'_{\bar{L},\theta} \\
= {}& -q^{-4}f'\overline{g_{+,3/0,3-}} - q^{-1}f_{+}\overline{g_{+,3/0,3}} + q^{-4}f_{3/0}\overline{g_{+3-}} \\
&- q^{-2}f_{3}\overline{g_{+,3/0,-}} + q^{-5}f_{-}\overline{g_{3/0,3-}} - qf_{+3}\overline{g_{+,3/0}} \\
&+ q^{-1}f_{+,3/0}\overline{g_{+3}} + q^{-2}f_{+-}\overline{g_{3/0,3}} \\
&+ (q^{-1} - q^{-3})f_{3/0,3}\overline{g_{3/0,3}} - q^{-2}f_{3/0,3}\overline{g_{+-}} - q^{-5}f_{3/0,-}\overline{g_{3-}}
\end{aligned}
$$

$$
\begin{aligned}
&+ q^{-3} f_{3-}\overline{g_{3/0,-}} - q f_{+,3/0,3}\overline{g_{+}} + f_{+3-}\overline{g_{3/0}} \\
&+ q^{-3} f_{3/0,3-}\overline{g_{-}} + f_{+,3/0,3-}\overline{g'},
\end{aligned} \tag{278}
$$

$$
\begin{aligned}
&\big\langle f, g\big\rangle'_{R,\theta} \\
&\quad = -q^{-4} f'\overline{g_{+3,3/0,-}} - q^{-1} f_{+}\overline{g_{+3,3/0,}} - q^{-4} f_{3/0}\overline{g_{+3-}} \\
&\qquad + q^{-2} f_{3}\overline{g_{+,3/0,-}} + q^{-5} f_{-}\overline{g_{3,3/0,-}} + q f_{+3}\overline{g_{+,3/0}} \\
&\qquad - q^{-1} f_{+,3/0}\overline{g_{+3}} + q^{-2} f_{+-}\overline{g_{3,3/0}} + (q^{-1} - q^{-3}) f_{3,3/0}\overline{g_{3,3/0}} \\
&\qquad - q^{-2} f_{3,3/0}\overline{g_{+-}} + q^{-5} f_{3/0,-}\overline{g_{3-}} - q^{-3} f_{3-}\overline{g_{3/0,-}} \\
&\qquad - q f_{+3,3/0}\overline{g_{+}} + q^{-2} f_{+,3/0,-}\overline{g_{3}} - f_{+3-}\overline{g_{3/0}} \\
&\qquad + q^{-3} f_{3,3/0,-}\overline{g_{-}} + f_{+3,3/0,-}\overline{g'},
\end{aligned} \tag{279}
$$

$$
\begin{aligned}
&\big\langle f, g\big\rangle'_{\bar{R},\theta} \\
&\quad = -q^{4} f'\overline{g_{-3,3/0,+}} - q f_{-}\overline{g_{-3,3/0,}} - q^{4} f_{3/0}\overline{g_{-3+}} + q^{2} f_{3}\overline{g_{-,3/0,+}} \\
&\qquad + q^{5} f_{+}\overline{g_{3,3/0,+}} + q^{-1} f_{-3}\overline{g_{-,3/0}} - q f_{-,3/0}\overline{g_{-3}} + q^{2} f_{-+}\overline{g_{3,3/0}} \\
&\qquad + (q - q^{3}) f_{3,3/0}\overline{g_{3,3/0}} - q^{2} f_{3,3/0}\overline{g_{-+}} + q^{5} f_{3/0,+}\overline{g_{3+}} \\
&\qquad - q^{3} f_{3+}\overline{g_{3/0,+}} - q^{-1} f_{-3,3/0}\overline{g_{-}} + q^{2} f_{-,3/0,+}\overline{g_{3}} - f_{-3+}\overline{g_{3/0}} \\
&\qquad + q^{3} f_{3,3/0,+}\overline{g_{+}} + f_{-,3/0,+}\overline{g'}.
\end{aligned} \tag{280}
$$

The sesquilinear forms defined by (260) again behave like scalars. Thus, many results about sesquilinear forms on symmetrized quantum spaces (see for example the reasonings about adjoint operators, invariance properties of sesquilinear forms, and Fourier-Plancherel identities in Part I of this paper) carry over to antismmetrized quantum spaces. The corresponding relations are again obtained by applying the above mentioned substitutions. In this regard it should also be mentioned that the constants κ_A have to be specified as follows:

(i) (antisymmetrized quantum plane)

$$
\kappa = \kappa_{\bar{L}} = \kappa_R = (\kappa_L)^{-1} = (\kappa_{\bar{R}})^{-1} = q^3, \tag{281}
$$

(ii) (antisymmetrized three-dimensional Euclidean space)

$$
\kappa = \kappa_{\bar{L}} = \kappa_R = (\kappa_L)^{-1} = (\kappa_{\bar{R}})^{-1} = -q^{-6}, \tag{282}
$$

(iii) (antisymmetrized four-dimensional Euclidean space)

$$\kappa = \kappa_{\bar{L}} = \kappa_R = (\kappa_L)^{-1} = (\kappa_{\bar{R}})^{-1} = q^{-4}, \tag{283}$$

(iv) (antisymmetrized q-deformed Minkowski space)

$$\kappa = \kappa_{\bar{L}} = \kappa_R = (\kappa_L)^{-1} = (\kappa_{\bar{R}})^{-1} = q^{4}. \tag{284}$$

Now, we come to eigenfunctions of position and momentum operators on antisymmetrized quantum spaces. These functions are respectively characterized by the equations

$$\begin{aligned} \theta^i \overset{\theta}{\cdot} u_\eta(\theta^j) &\sim u_\eta(\theta^j) \overset{\eta}{\cdot} \eta^i, \\ \bar{u}_\eta(\theta^j) \overset{\theta}{\cdot} \theta^i &\sim \eta^i \overset{\eta}{\cdot} \bar{u}_\eta(\theta^j), \end{aligned} \tag{285}$$

and

$$\begin{aligned} \mathrm{i}\partial^k \overset{\theta}{\triangleright} u_\rho(\theta^j) &= u_\rho(\theta^j) \overset{\rho}{\cdot} \rho^k, \\ \bar{u}_\rho(\theta^j) \overset{\theta}{\triangleleft} (\mathrm{i}\partial^k) &= \rho^k \overset{\rho}{\cdot} \bar{u}_\rho(\theta^j). \end{aligned} \tag{286}$$

Solutions to these equations are given by

$$\begin{aligned} (u_A)_\eta(\theta^i) &\equiv (\mathrm{vol}_A)^{-1}\, \delta^n_A(\theta^i \oplus_A (\ominus_A\, \kappa_A \eta^j)), \\ (\bar{u}_A)_\eta(\theta^i) &\equiv (\mathrm{vol}_A)^{-1}\, \delta^n_A((\ominus_A\, \kappa_A \eta^j) \oplus_A \theta^i), \end{aligned} \tag{287}$$

and

$$\begin{aligned} (u_{\bar{R},L})_\rho(\theta^i) &\equiv (\mathrm{vol}_L)^{-1/2} \exp(\theta^i|\mathrm{i}^{-1}\rho^k)_{\bar{R},L}, \\ (u_{R,\bar{L}})_\rho(\theta^i) &\equiv (\mathrm{vol}_{\bar{L}})^{-1/2} \exp(\theta^i|\mathrm{i}^{-1}\rho^k)_{R,\bar{L}}, \end{aligned} \tag{288}$$

$$\begin{aligned} (\bar{u}_{\bar{R},L})_\rho(\theta^i) &\equiv (\mathrm{vol}_{\bar{R}})^{-1/2} \exp(\mathrm{i}^{-1}\rho^k|\theta^i)_{\bar{R},L}, \\ (\bar{u}_{R,\bar{L}})_\rho(\theta^i) &\equiv (\mathrm{vol}_{R})^{-1/2} \exp(\mathrm{i}^{-1}\rho^k|\theta^i)_{R,\bar{L}}, \end{aligned} \tag{289}$$

where for the volume elements we have to take the expressions in (258) and (259).

Again, these eigenfunctions fulfill completeness and orthonormality relations. Their explicit form can be read off from the results in Sec. 3.2 and 3.3. We would like to illustrate this by an example. For example, the

identities (88) and (97) correspond to

$$
\begin{aligned}
&\int d_L^n \eta \, (u_{\bar{R}})_\eta (\tilde{\theta}^i) \overset{\eta}{\cdot} (\bar{u}_{\bar{R}})_\eta (\theta^k) \\
&\qquad = (-1)^n \big\langle (\bar{u}_L)_\eta (\tilde{\theta}^i), (\bar{u}_{\bar{R}})_\eta (\theta^k) \big\rangle_{L,\eta} \\
&\qquad = (\mathrm{vol}_{\bar{R}})^{-1} \delta_{\bar{R}}^n (\tilde{\theta}^i \oplus_{\bar{R}} (\ominus_{\bar{R}} \, \kappa^{-1} \theta^k)),
\end{aligned}
\tag{290}
$$

and

$$
\begin{aligned}
&\int d_L^n \theta \, (\bar{u}_{\bar{R}})_{\tilde{\eta}} (\theta^j) \overset{\theta}{\cdot} (u_{\bar{R}})_\eta (\theta^l) \\
&\qquad = (-1)^n \big\langle (\bar{u}_{\bar{R}})_{\tilde{\eta}} (\theta^j), (\bar{u}_L)_\eta (\theta^l) \big\rangle'_{L,\theta} \\
&\qquad = (\mathrm{vol}_{\bar{R}})^{-1} \delta_{\bar{R}}^n (\tilde{\eta}^i \oplus_{\bar{R}} (\ominus_{\bar{R}} \, \kappa^{-1} \eta^k)),
\end{aligned}
\tag{291}
$$

The operators associated with Grassmann position and Grassmann momentum act on supernumbers as

$$
\mathcal{N}^k \overset{\theta}{\triangleright} f(\theta^j) = \theta^k \overset{\theta}{\cdot} f(\theta^j), \qquad f(\theta^j) \overset{\theta}{\cdot} \theta^k = f(\theta^j) \overset{\theta}{\triangleleft} \mathcal{N}^k,
\tag{292}
$$

and

$$
\begin{aligned}
&\mathcal{P}^k \overset{\theta}{\triangleright} f(\theta^j) = \mathrm{i}\partial^k \overset{\theta}{\triangleright} f(\theta^j), &&\hat{\mathcal{P}}^k \overset{\theta}{\triangleright} f(\theta^j) = \mathrm{i}\hat{\partial}^k \overset{\theta}{\triangleright} f(\theta^j), \\
&\mathcal{P}^k \overset{\theta}{\bar{\triangleright}} f(\theta^j) = \mathrm{i}\partial^k \overset{\theta}{\bar{\triangleright}} f(\theta^j), &&\hat{\mathcal{P}}^k \overset{\theta}{\bar{\triangleright}} f(\theta^j) = \mathrm{i}\hat{\partial}^k \overset{\theta}{\bar{\triangleright}} f(\theta^j),
\end{aligned}
\tag{293}
$$

$$
\begin{aligned}
&f(\theta^j) \overset{\theta}{\triangleleft} \mathcal{P}^k = f(\theta^j) \overset{\theta}{\triangleleft} (\mathrm{i}\partial^k), &&f(\theta^j) \overset{\theta}{\triangleleft} \hat{\mathcal{P}}^k = f(\theta^j) \overset{\theta}{\triangleleft} (\mathrm{i}\hat{\partial}^k), \\
&f(\theta^j) \overset{\theta}{\bar{\triangleleft}} \mathcal{P}^k = f(\theta^j) \overset{\theta}{\bar{\triangleleft}} (\mathrm{i}\partial^k), &&f(\theta^j) \overset{\theta}{\bar{\triangleleft}} \hat{\mathcal{P}}^k = f(\theta^j) \overset{\theta}{\bar{\triangleleft}} (\mathrm{i}\hat{\partial}^k),
\end{aligned}
\tag{294}
$$

Their matrix representations are obtained from the results in Sec. 3.4 by the very same method that enables us to find the completeness and orthonormality relations for Grassmann eigenfunctions. For example, we find from (135) and (147) that

$$
\begin{aligned}
(\mathcal{N}'_{\bar{L}})^m_{\tilde{\eta}\eta} &= (-1)^n \big\langle (\bar{u}_R)_{\tilde{\eta}} (\theta^l) \overset{\theta}{\cdot} \theta^m, (\bar{u}_{\bar{L}})_\eta (\theta^r) \big\rangle'_{\bar{L},\theta} \\
&= (\mathrm{vol}_R)^{-1} \tilde{\eta}^m \overset{\tilde{\eta}}{\cdot} \delta_R^n (\tilde{\eta}^j \oplus_R (\ominus_R \, \kappa \eta^i)) \\
&= (\mathrm{vol}_R)^{-1} \delta_R^n ((\ominus_R \, \kappa \tilde{\eta}^j) \oplus_R \eta^i) \overset{\tilde{\eta}}{\cdot} \eta^m,
\end{aligned}
\tag{295}
$$

and

$$(\hat{\mathcal{P}}'_{\bar{L}})^m_{\tilde{\eta}\eta} = (-1)^n \big\langle (\bar{u}_R)_{\tilde{\eta}}(\theta^l) \overset{\theta}{\triangleleft} \hat{P}^m, (\bar{u}_{\bar{L}})_\eta(\theta^r) \big\rangle'_{\bar{L},\theta}$$
$$= (\text{vol}_R)^{-1} \text{i}\hat{\partial}^m \overset{\tilde{\eta}}{\triangleright} \delta^n_R(\tilde{\eta}^j \oplus_R (\ominus_R \, \kappa\eta^i))$$
$$= (\text{vol}_R)^{-1} \delta^n_R(\tilde{\eta}^j \oplus_R (\ominus_R \, \kappa\eta^i)) \overset{\eta}{\triangleleft} (\text{i}\hat{\partial}^m), \tag{296}$$

Expectation values and probability densities can be defined in complete analogy to the reasonings in Sec. 5. Thus, the details are left to the reader.

Our last comment in this section concerns the canonical commutation relations for the operators $\mathcal{N}^i$ and $\mathcal{P}^i$. The components $\mathcal{N}^i$ satisfy among each other the commutation relations for q-deformed Grassmann variables (for their explicit form see for example Ref. [24]) and the same holds for the operators $\mathcal{P}^i$. Again, the commutation relations between $\mathcal{P}^i$ and $\mathcal{N}^i$ are nothing other than the q-deformed versions of antisymmetrized Leibniz rules:

$$\mathcal{P}^k \mathcal{N}^l - k(\hat{R}^{-1})^{kl}_{mn} \mathcal{N}^m \mathcal{P}^n = \text{i}g^{kl}, \tag{297}$$

or

$$\mathcal{P}^k \mathcal{N}^l - k^{-1} \hat{R}^{kl}_{mn} \mathcal{N}^m \mathcal{P}^n = i\bar{g}^{kl}. \tag{298}$$

The values for k are now given by

(i) (quantum plane) $k = 1$,

(ii) (three-dimensional Euclidean space) $k = q^{-4}$,

(iii) (four-dimensional Euclidean space) $k = q^{-1}$,

(iv) (q-deformed Minkowski space) $k = q^{-1}$.

7 Conclusion

Let us conclude our reasonings by some remarks. Within the mathematical framework we completed in Part I of this article by introducing Fourier transformations and sesquilinear forms on q-deformed quantum spaces we developed basic concepts of quantum kinematics on q-deformed quantum spaces. This task could be achieved in complete analogy to the classical situation. As a consequence of this observation we should regain the classical theory when the value of the deformation parameter q tends to 1.

To give our formalism a physical meaning we introduced the idea of so-called quasipoints. Such quasipoints can be considered as elements of

certain vector spaces. In these vector spaces we were able to identify q-analogs of momentum and position eigenfunctions. Using the results about q-deformed Fourier transformations in Part I we showed that these eigenfunctions establish orthonormal bases in q-deformed position or momentum space. In analogy to the undeformed case physical wave packets are characterized by normalization conditions formulated by means of symmetrical sesquilinear forms. The observation that wave packets can be expanded in terms of position or momentum eigenfunctions enabled us to introduce transition probabilities and expectation values in a consistent way. With these examinations we laid the foundations for describing free particles in q-deformed quantum mechanics.

However, there is one essential difference between the deformed and the undeformed theory. As already mentioned in Part I of this article, we can distinguish two geometries that transform into each other via the operation of conjugation [51,53,54]. These geometries correspond to the two categories characterized by the braidings Ψ and Ψ^{-1}. In a q-deformed theory we have to combine both geometries to get real quantities. In the undeformed case this problem does not arise, since both geometries become identical in that limit.

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A Quantum spaces

In this appendix we provide some key notation for the quantum spaces we are interested in for physical reasons, i.e. Manin plane, q-deformed Euclidean space in three and four dimensions, and q-deformed Minkowski space. For each case we give the defining relations, the quantum metric, and the conjugation properties.

The coordinates of the two-dimensional q-deformed quantum plane fulfill the relation [55, 56]

$$X^1X^2 = qX^2X^1, \tag{299}$$

whereas the quantum metric is given by a matrix ε^{ij} with non-vanishing elements

$$\varepsilon^{12} = q^{-1/2}, \quad \varepsilon^{21} = -q^{1/2}. \tag{300}$$

The relation (299) is compatible with the conjugation assignment

$$\overline{X^i} = -\varepsilon_{ij} X^j, \tag{301}$$

where ε_{ij} denotes the inverse of ε^{ij}.

The commutation relations for the q-deformed Euclidean space in three dimensions read [31]

$$\begin{aligned} X^3 X^+ &= q^2 X^+ X^3, \\ X^- X^3 &= q^2 X^3 X^-, \\ X^- X^+ &= X^+ X^- + \lambda X^3 X^3. \end{aligned} \tag{302}$$

The non-vanishing elements of the quantum metric are

$$g^{+-} = -q, \quad g^{33} = 1, \quad g^{-+} = -q^{-1}. \tag{303}$$

The conjugation properties of coordinates are given by

$$\overline{X^A} = g_{AB} X^B, \tag{304}$$

with g_{AB} denoting the inverse of g^{AB}. If we are looking for coordinates subject to $\overline{Y^i} = Y^i$ we can choose

$$\begin{aligned} Y^1 &= \frac{\mathrm{i}}{q^{1/2} + q^{-1/2}} (q^{-1/2} X^+ + q^{1/2} X^-), \\ Y^2 &= \frac{1}{q^{1/2} + q^{-1/2}} (q^{-1/2} X^+ - q^{1/2} X^-), \\ Y^3 &= X^3. \end{aligned} \tag{305}$$

For the four-dimensional Euclidean space we have the relations [27, 57]

$$\begin{aligned} X^1 X^2 &= q X^2 X^1, \\ X^1 X^3 &= q X^3 X^1, \\ X^3 X^4 &= q X^4 X^3, \\ X^2 X^4 &= q X^4 X^2, \\ X^2 X^3 &= X^3 X^2, \\ X^4 X^1 &= X^1 X^4 + \lambda X^2 X^3. \end{aligned} \tag{306}$$

The non-vanishing components of the corresponding quantum metric read

$$g^{14} = q^{-1}, \quad g^{23} = g^{32} = 1, \quad g^{41} = q. \tag{307}$$

If g_{ij} again denotes the inverse of g^{ij} it holds

$$\overline{X^i} = g_{ij} X^j. \tag{308}$$

Using this relation it is easy to check that the following independent coordinates are invariant under conjugation [58]:

$$\begin{aligned}
Y^1 &= \frac{1}{q^{1/2} + q^{-1/2}} (q^{1/2} X^1 + q^{-1/2} X^4), \\
Y^2 &= \frac{1}{2} (X^2 + X^3), \\
Y^3 &= \frac{\mathrm{i}}{2} (X^2 - X^3), \\
Y^4 &= \frac{\mathrm{i}}{q^{1/2} + q^{-1/2}} (q^{1/2} X^1 - q^{-1/2} X^4).
\end{aligned} \tag{309}$$

The coordinates of q-deformed Minkowski space obey the relations [27]

$$\begin{aligned}
X^\mu X^0 &= X^0 X^\mu, \quad \mu \in \{0, +, -, 3\}, \\
X^- X^3 - q^2 X^3 X^- &= -q\lambda X^0 X^-, \\
X^3 X^+ - q^2 X^+ X^3 &= -q\lambda X^0 X^+, \\
X^- X^+ - X^+ X^- &= \lambda (X^3 X^3 - X^0 X^3).
\end{aligned} \tag{310}$$

As non-vanishing components of the corresponding metric we have

$$\eta^{00} = -1, \quad \eta^{33} = 1, \quad \eta^{+-} = -q, \quad \eta^{-+} = -q^{-1}. \tag{311}$$

(For other deformations of Minkowski spacetime we refer to Refs. [59–65].) The conjugation on q-deformed Minkowski space is determined by

$$\overline{X^0} = X^0, \quad \overline{X^3} = X^3, \quad \overline{X^\pm} = -q^{\mp 1} X^\mp. \tag{312}$$

A set of independent coordinates being invariant under conjugation is now given by $Y^0 = X^0$ and the coordinates introduced in (305).

References

[1] M. Fichtenmüller, A. Lorek and J. Wess, *q-deformed Phase Space and its Lattice Structure,* Z. Phys. C **71** (1996) 533, [hep-th/9511106].

[2] B. L. Cerchiai and J. Wess, *q-Deformed Minkowski Space based on a q-Lorentz Algebra,* Eur. Phys. J. C **5** (1998) 553, [math.qa/9801104].

[3] J. Schwinger, *Quantum Electrodynamics*, Dover, New York, 1958.

[4] W. Heisenberg, *Über die in der Theorie der Elementarteilchen auftretende universelle Länge,* Ann. Phys. **32** (1938) 20.

[5] H. Grosse, C. Klimčik and P. Prešnajder, *Towards finite quantum field theory in non-commutative geometry,* Int. J. Theor. Phys. **35** (1996) 231, [hep-th/9505175].

[6] S. Majid, *On the q-regularisation,* Int. J. Mod. Phys. A **5** (1990) 4689.

[7] R. Oeckl, *Braided Quantum Field Theory,* Commun. Math. Phys. **217** (2001) 451.

[8] C. Blohmann, *Free q-deformed relativistic wave equations by representation theory*, Eur. Phys. J. C **30** (2003) 435, [hep-th/0111172].

[9] P. P. Kulish and N. Y. Reshetikin, *Quantum linear problem for the Sine-Gordon equation and higher representations,* J. Sov. Math. **23** (1983) 2345.

[10] S. L. Woronowicz, *Compact matrix pseudogroups,* Commun. Math. Phys. **111** (1987) 613.

[11] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation,* Sov. Math. Dokl. **32** (1985) 254.

[12] M. Jimbo, *A q-analogue of U(g) and the Yang-Baxter equation,* Lett. Math. Phys. **10** (1985) 63.

[13] V. G. Drinfeld, *Quantum groups*, in A. M. Gleason, ed., Proceedings of the International Congress of Mathematicians, Amer. Math. Soc., 798 (1986).

[14] N. Yu. Reshetikhin, L. A. Takhtadzhyan and L. D. Faddeev, *Quantization of Lie Groups and Lie Algebras*, Leningrad Math. J. **1** (1990) 193.

[15] M. Takeuchi, *Matrix Bialgebras and Quantum Groups,* Israel J. Math. **72** (1990) 232.

[16] I. M. Gelfand and M. A. Naimark, *On the embedding of normed linear rings into the ring of operators in Hilbert space*, Math. Sbornik **12** (1943) 197.

[17] H. Hopf, *Über die Topologie der Gruppenmannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. Math. **42** (1941) 22.

[18] H. Wachter and M. Wohlgenannt, $*$*-Products on quantum spaces,* Eur. Phys. J. C **23** (2002) 761, [hep-th/0103120].

[19] C. Bauer and H. Wachter, *Operator representations on quantum spaces,* Eur. Phys. J. C **31** (2003) 261, [math-ph/0201023].

[20] H. Wachter, *q-Integration on quantum spaces,* Eur. Phys. J. C **32** (2004) 281, [hep-th/0206083].

[21] H. Wachter, *q-Exponentials on quantum spaces,* Eur. Phys. J. C **37** (2004) 379, [hep-th/0401113].

[22] H. Wachter, *q-Translations on quantum spaces,* preprint, [hep-th/0410205].

[23] H. Wachter, *Braided products for quantum spaces,* preprint, [math-ph/0509018].

[24] D. Mikulovic, A. Schmidt and H. Wachter, *Grassmann variables on quantum spaces,* Eur. Phys. J. C **45** (2006) 529, [hep-th/0407273].

[25] A. Schmidt and H. Wachter, *Superanalysis on quantum spaces*, JHEP **0601** (2006) 84, [hep-th/0411180].

[26] A. Schmidt and H. Wachter, *q-Deformed quantum Lie algebras*, J. Geom. Phys. **56** (2006) 2289, [math-ph/0509032].

[27] U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, *Tensor Representations of the Quantum Group* $SL_q(2)$ *and Quantum Minkowski Space,* Z. Phys. C **48** (1990) 159.

[28] P. Podleś and S. L. Woronowicz, *Quantum Deformation of Lorentz Group*, Commun. Math. Phys. **130** (1990) 381.

[29] W. B. Schmidke, J. Wess and B. Zumino, *A q-deformed Lorentz Algebra in Minkowski phase space,* Z. Phys. C **52** (1991) 471.

[30] S. Majid, *Examples of braided groups and braided matrices,* J. Math. Phys. **32** (1991) 3246.

[31] A. Lorek, W. Weich and J. Wess, *Non-commutative Euclidean and Minkowski Structures,* Z. Phys. C **76** (1997) 375, [q-alg/9702025].

[32] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplane*, Nucl. Phys. B. Suppl. **18** (1991) 302.

[33] U. Carow-Watamura, M. Schlieker and S. Watamura, *$SO_q(N)$-covariant differential calculus on quantum space and deformation of Schrödinger equation,* Z. Phys. C **49** (1991) 439.

[34] X. C. Song, *Covariant differential calculus on quantum minkowski space and q-analog of Dirac equation,* Z. Phys. C **55** (1992) 417.

[35] O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, *q-Deformed Poincaré Algebra,* Commun. Math. Phys. **150** (1992) 495.

[36] S. Majid, *Braided momentum in the q-Poincaré group*, J. Math. Phys. **34** (1993) 2045.

[37] H. Wachter, *Analysis on q-deformed quantum spaces*, to appear in Int. J. Mod. Phys. A, [math-ph/0604028].

[38] S. Majid, *Representations, duals and quantum doubles of monoidal categories,* Suppl. Rend. Circ. Mat. Palermo, Ser. II, **26** (1991) 197.

[39] S. Majid, *Algebras and Hopf Algebras in Braided Categories,* Lec. Notes Pure Appl. Math. **158** (1994) 55.

[40] S. Majid, *Introduction to Braided Geometry and q-Minkowski Space,* preprint (1994), [hep-th/9410241].

[41] S. Majid, *Beyond Supersymmetry and Quantum Symmetry (an itroduction to braided groups and braided matrices)*, in M. L. Ge and H. J. de Vega, eds., Quantum Groups, Integrable Statistical Models and Knot Theory, World Scientific, 231 (1993).

[42] S. Majid, *Free braided differential calculus, braided binomial theorem and the braided exponential map,* J. Math. Phys. **34** (1993) 4843.

[43] S. Majid, *Foundations of Quantum Group Theory,* University Press, Cambridge (1995).

[44] U. Meyer, *q-Lorentz group and braided coaddition on q-Minkowski space*, Commun. Math. Phys. **168** (1995) 249.

[45] C. Chryssomalakos, P. Schupp, and P. Watts, *Translations, Integrals And Fourier Transforms In The Quantum Plane*, in A. Ali, J. Ellis and S. Randjbar-Daemi, eds., Salamfestschrift, Proceedings of the Conference on Highlights of Particle and Condensed Matter Physics, ICTP, Trieste (1993).

[46] A. Schirrmacher, *Generalized q-exponentials related to orthogonal quantum groups and Fourier transformations of noncommuative spaces*, J. Math. Phys. **36** (3) (1995) 1531.

[47] B. L. Cerchiai, R. Hinterding, J. Madore and J. Wess, *A Calculus Based on a q-deformed Heisenberg Algebra,* Eur. Phys. J. C **8** (1999) 547, [math-ph/9809160].

[48] V. Kac and P. Cheung, *Quantum Calculus,* Springer Verlag, Berlin (2000).

[49] J. Wess, *q-deformed Heisenberg Algebras*, in H. Gausterer, H. Grosse and L. Pittner, eds., Proceedings of the 38. Internationale Universitätswochen für Kern- und Teilchenphysik, no. 543 in Lect. Notes in Phys., Springer-Verlag, Schladming (2000), [math-ph/9910013].

[50] A. Kempf and S. Majid, *Algebraic q-integration and Fourier theory on quantum and braided spaces,* J. Math. Phys. **35** (1994) 6802.

[51] S. Majid, *Quasi-$*$-structure on q-Poincaré algebras*, J. Geom. Phys. **22** (1997) 14.

[52] F. H. Jackson, *q-Integration,* Proc. Durham Phil. Soc. **7** (1927) 182.

[53] O. Ogievetsky and B. Zumino, *Reality in the differential calculus on q-Euclidean spaces*, Lett. Math. Phys. **25** (1992) 121.

[54] S. Majid, *$*$-structures on braided spaces*, J. Math. Phys. **36** (1995) 4436.

[55] Y. I. Manin, *Quantum Groups and Non-Commutative Geometry*, Centre de Recherche Mathématiques, Montreal (1988).

[56] M. Schlieker and W. Scholl, *Spinor calculus for quantum groups*, Z. Phys. C **47** (1990) 625.

[57] A. Klimyk and K. Schmüdgen, *Quantum Groups and their Representations,* Springer Verlag, Berlin (1997).

[58] H. Ocampo, $SO_q(4)$ *quantum mechanics*, Z. Phys. C **70** (1996) 525.

[59] J. Lukierski, A. Nowicki and H. Ruegg, *New Quantum Poincaré Algebra and κ-deformed Field Theory,* Phys. Lett. B **293** (1992) 344.

[60] L. Castellani, *Differential Calculus on $ISO_q(N)$, Quantum Poincaré Algebra and q-Gravity,* preprint, [hep-th/9312179].

[61] V. K. Dobrev, *New q-Minkowski space-time and q-Maxwell equations hierarchy from q-conformal invariance,* Phys. Lett. B **341** (1994) 133.

[62] S. Doplicher, K. Fredenhagen and J. E. Roberts, *The Quantum Structure of Space-Time at the Planck Scale and Quantum fields*, Commun. Math. Phys. **172** (1995) 187.

[63] M. Chaichian and A. P. Demichev, *Quantum Poincaré group without dilatation and twisted classical algebra*, J. Math. Phys. **36** (1995) 398.

[64] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, *On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and its Implications on Noncommutative QFT,* Phys. Lett. B **604** (2004) 98, [hep-th/0408062].

[65] F. Koch and E. Tsouchnika, *Construction of θ-Poincaré Algebras and their invariants on $\mathcal{M}_\theta$*, Nucl. Phys. B **717** (2005) 387, [hep-th/0409012].